# Bearing Accuracy against Hard Targets with SeaSonde DF Antennas

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**Significant Result:** All radar systems that attempt to determine bearing of a target are limited in angular accuracy by the signal-to-noise ratio (SNR). The higher the SNR, the more accurately bearing can be estimated, regardless of the type of antenna system (beam forming or direction finding).

There are three ways to establish the exact relationship: (i) theoretical derivation; (ii) Monte-Carlo simulations; (iii) experimental measurement. We have done all three, and they are all in agreement.

The theoretical derivation is presented below. This is then compared with simulations, where the two methods are shown to agree. Finally, results are presented later emphasizing the improvement in accuracy obtained from using measured antenna patterns; these also exhibit the expected SNR dependence predicted by theory and simulations.

The theoretical result derived here for a simple, single-angle arctangent DF algorithm is:

$$\square_B = \frac{40 \text{ degrees}}{\sqrt{SNR}}$$

where SNR is measured in terms of absolute power (not decibels), as the difference in levels between the target peak and the surrounding spectral noise.

We use the MUSIC algorithm for direction finding with SeaSonde processing rather than the simple Atan2 algorithm, because the latter only works for ideal sine-cosine loop patterns rather than the more general distorted patterns encountered and measured in practice. It is not possible to derive a theoretical closed-form result like that above for MUSIC, even for perfect, idealized patterns. However, our simulations show that the two results for accuracy are identical (i.e., Atan2 and MUSIC) when perfect idealized antenna patterns apply.

**Derivation:** The Atan2 algorithm differs from the standard ArcTan algorithm in that the former is able to resolve angle in all quadrants over 360°, while the latter has a 180° ambiguity. The monopole of the three-element SeaSonde antenna unit provides the phase reference needed for the Atan2 algorithm that avoids ambiguity.

Expressed in FORTRAN or MATLAB, the Atan2 algorithm would take the form:

$$\square = \text{Atan2}\left(\frac{V_2}{V_3}, \frac{V_1}{V_3}\right)$$

where $v_1$ is the voltage received by Loop #1 (which we assume for convenience to be a cosine pattern; $v_2$ is the voltage from Loop #2 assumed to be a sine function of bearing angle; and $v_3$ is the voltage received on the monopole, assumed to be omni-directional.

For a signal coming from a target at bearing angle $\varphi$, these voltages can be expressed as shown below. The first term is the unity-normalized signal from the target, and the second term is the noise. (I.e., the signal normalization is such that its amplitude is unity.) The added noise is assumed to be a complex, zero-mean Gaussian random variable. At HF, this comes from outside the antenna, originating from atmospheric sources (e.g., thunderstorms worldwide). The most reasonable noise model to assume -- in the absence of other site-specific information -- is an isotropic distribution with bearing angle. In this model, the noise from each bearing direction is uncorrelated with that from all other bearings.

$$v_1 = \cos\varphi + n_1; \quad n_1 \equiv \frac{1}{2\pi}\int_0^{2\pi} \nu(\varphi')\cos\varphi' \, d\varphi'$$

$$v_2 = \sin\varphi + n_2; \quad n_2 \equiv \frac{1}{2\pi}\int_0^{2\pi} \nu(\varphi')\sin\varphi' \, d\varphi'$$

$$v_3 = 1 + n_3; \quad n_3 \equiv \frac{1}{2\pi}\int_0^{2\pi} \nu(\varphi') \, d\varphi'$$

Although the noise mean is zero, its variance can be represented as an infinite ensemble average on the omni-directional monopole as follows:

$$\langle n_3 n^*_3 \rangle \equiv \frac{1}{(2\pi)^2}\int_0^{2\pi} d\varphi'' \int_0^{2\pi} d\varphi' \langle \nu(\varphi')\nu^*(\varphi'') \rangle$$

We now define an angular spectral distribution, $S(\varphi)$, for the noise as follows, where as stated previously, we take the spectrum to be isotropic (omni-directional, or uniform over all angles):

$$\langle \nu(\varphi')\nu^*(\varphi'') \rangle \equiv 2\pi S(\varphi' - \varphi'')\delta(\varphi' - \varphi'') = 2\pi\sigma_n^2\delta(\varphi' - \varphi'')$$

which defines the noise power or variance on the monopole below. Likewise, we define the noise powers or variances on the loops.

$$\langle n_3 n^*_3 \rangle = \sigma_n^2;$$

$$\langle n_1 n^*_1 \rangle = \frac{\sigma_n^2}{2\pi}\int_0^{2\pi} \cos^2\varphi' \, d\varphi' = \frac{\sigma_n^2}{2}; \quad \langle n_2 n^*_2 \rangle = \frac{\sigma_n^2}{2\pi}\int_0^{2\pi} \sin^2\varphi' \, d\varphi' = \frac{\sigma_n^2}{2}$$

Using these definitions and assumptions, we can show why the noise signals among the three antenna elements are uncorrelated:

$$\left\langle n_1 n^*_3 \right\rangle = \frac{\sigma_n^2}{2\pi}\int_0^{2\pi} \cos\varphi' d\varphi' = 0; \quad \left\langle n_2 n^*_3 \right\rangle = \frac{\sigma_n^2}{2\pi}\int_0^{2\pi} \sin\varphi' d\varphi' = 0$$

and

$$\left\langle n_1 n^*_2 \right\rangle = \frac{\sigma_n^2}{2\pi}\int_0^{2\pi} \cos\varphi' \sin\varphi' d\varphi' = 0 .$$

For the sake of deriving the bearing angle variance for this noise model, we will use Euler's formula to analyze the Atan2's DF performance. This is given by:

$$e^{i(\varphi + \delta)} = \frac{v_1}{v_3} + i\frac{v_2}{v_3}$$

where $\delta$ is the random fluctuation in the target bearing angle $\varphi$ due to the noise.

The Atan2 function serves divide out the monopole voltage and its fluctuations, except for the phase referencing. Therefore, one can express this exponential as:

$$e^{i(\varphi + \delta)} = \left(\cos\varphi + n_1\right) + i\left(\sin\varphi + n_2\right),$$

which can be re-written:

$$e^{i(\varphi + \delta)} \cong e^{i\varphi} + n_1 + i n_2$$

Next, divide the exponential part containing, $e^{i\varphi}$ from both sides. Then, because the noise fluction portion of the exponential is small, it can be expanded in its first two terms as:

$$e^{i\delta} \cong \left(1 + i\delta\right) = 1 + e^{-i\varphi}\left(n_1 + i n_2\right),$$

giving an expression for the small noise fluctuation contribution to the bearing as:

$$\delta = -i e^{-i\varphi}\left(n_1 + i n_2\right).$$

The variance of this quantity can now be taken:

$$\left\langle \square^2 \right\rangle = \left\langle \left[ -\, i\, e^{-i\square} \left( n_1 + i\, n_2 \right) \right] \left[ -\, i\, e^{-i\square} \left( n_1 + i\, n_2 \right) \right]^* \right\rangle ,$$

which, when expanded -- and use is made of the above averaging relations -- reduces to two non-vanishing variance terms:

$$\left\langle \square^2 \right\rangle = \left\langle n_1 n_1^* \right\rangle + \left\langle n_2 n_2^* \right\rangle = \square_n^2 .$$

Thus we have that the bearing variance (in radians-squared) is directly equal to the noise variance on the monopole (normalized by the signal amplitude).

We can relate this to the signal-to-noise ratio and define the bearing standard deviation (or rms error) as:

$$\square_B \equiv \sqrt{\left\langle \square^2 \right\rangle} = \frac{1}{\sqrt{SNR}}$$

Recall in this derivation, we took the target signal to be pure real; perfect loop and monopole patterns are expressible in terms of pure real quantities (sine, cosine, and unity), although patterns of actual antennas in practice are complex. However, the noise model we used here was complex.

Because the real and imaginary parts contribute equally to the random noise, the above formula can be modified to handle the situation where the noise is pure real by dividing the noise power entering the calculations by a factor of two. This gives rise to the formula that would be applicable when the Atan2 algorithm is used (because the Atan2 by definition only makes sense when all quantities are pure real). The latter formula is then:

$$\square_B = \frac{1}{\sqrt{2 \cdot SNR}} ,$$

where the units of bearing error in this and the previous equation (for complex noise) are radians.

Converted to degrees, a more useful representation therefore becomes:

$$\square_B = \frac{40 \text{ degrees}}{\sqrt{SNR}} ,$$

which is the formula given as the 'Significant Result' on the first page.

***Our Monte-Carlo simulations given as the PowerPoint slide show that indeed, this theoretically derived formula exactly describes the behavior of bearing error with noise. Furthermore, MUSIC direction finding when applied to the same signals plus***

***noise via Monte-Carlo simulations also exhibit this same behavior. Thus, three different analyses all lead us back to the same noise depencence. However, if there is a bearing bias, as for example, when the antenna pattern used with MUSIC does not represent the true situation, then a bearing-error offset is added to this inverse SNR dependence, as is shown subsequently.***