Effects of Shallow Water on Radar Measurements

In our previous memo on the physics of surface gravity waves, we derived and discussed solutions for the waveheight, velocity, and frequency -- and their relatiohsip to the wavelength. We demonstrated that there is a square-root relation between wave frequency or wave velocity and the wavelength. This is an unusual dispersion relation not found frequently in physics. But in shallow water, it gets even more interesting and complicated. We saw that there is a hyperbolic relationship thrown in that contains the water depth. In *very* shallow water, the velocity becomes constant, independent of the spatial wavelength. ***This paper explores further the impacts of these relations on HF radar echoes and extraction of current, waves, and tsunamis from them.***

1. <u>How Does Shallow Water Affect Current Measurement</u>? When no currents are present, the first-order Bragg-peak echoes for HF backscatter radars are two very narrow peaks. The Bragg waves that produced them were sinusoids in the time domain, and these give rise to echo signals whose time domain voltage representations are sinusoids also, proportional to the heights as given by Eq. (25) of our previous memo. But note: this was the "deep-water" expression. ***All CODAR (as well as WERA, OSCR, etc.) processing up to now has assumed deep water,*** for good reason. This has been adequate in almost all circumstances. But not everywhere, especially as we move to lower frequencies to get longer ranges (e.g., 5 MHz).

Referring to Eq. (25), we can write the equation for the first-order echo in the time domain as:

$$s(x,t) = a_+ \cos\left(\frac{4\pi f_{MHz}}{300} x + \sqrt{\frac{g 4\pi f_{MHz}}{300}}\, t - \xi_+ \right) + a_- \cos\left(\frac{4\pi f_{MHz}}{300} x - \sqrt{\frac{g 4\pi f_{MHz}}{300}}\, t - \xi_- \right) \quad (1)$$

where x would be equivalent to range to the echo cell. The temporal transform of this signal (i.e., FFT) comprises two impulse functions (very narrow spikes) that we have called "Bragg peaks". In deep water, they always occur at the same positions defined by the square root of the radar frequency, symmetrically placed about DC (i.e., zero Doppler). Their amplitudes need not be the same, but depend on the dominant wave/wind direction. So, omitting the range or distance variable, the 2nd-FFT signal output becomes:

$$S_D(\omega) = a_+ \exp(i\xi_+)\partial\left(\omega - \sqrt{\frac{g4\pi f_{MHz}}{300}}\right) + a_- \exp(i\xi_-)\partial\left(\omega + \sqrt{\frac{g4\pi f_{MHz}}{300}}\right) \quad (2)$$

Currents are determined by measuring the shift of the Bragg peaks from their positions given above in Eq. (2), as the water is transported away from the radar with radial velocity v_r. Thus the counterparts to the above equations that include the current velocity toward the positive x-direction (in addition to the wave velocity) become:

$$s(x,t) = a_+ \cos\left(\frac{4\pi f_{MHz}}{300}x + \sqrt{\frac{g4\pi f_{MHz}}{300}}t + \frac{4\pi v_r f_{MHz}}{300}t - \xi_+\right) + a_- \cos\left(\frac{4\pi f_{MHz}}{300}x - \sqrt{\frac{g4\pi f_{MHz}}{300}}t + \frac{4\pi v_r f_{MHz}}{300}t - \xi_-\right) \quad (3)$$

and the signal spectrum with current present becomes:

$$S_D(\omega) = a_+ \exp(i\xi_+)\partial\left(\omega - \sqrt{\frac{g4\pi f_{MHz}}{300}} - \frac{4\pi v_r f_{MHz}}{300}\right) + a_- \exp(i\xi_-)\partial\left(\omega + \sqrt{\frac{g4\pi f_{MHz}}{300}} - \frac{4\pi v_r f_{MHz}}{300}\right) \quad (4)$$

So both "Bragg peaks" get shifted to the right by the amount $\frac{2v_r f_{MHz}}{300}$ Hz (the frequency units in the equations are all radians/second). Since the unshifted positions are always known and the same in deep water, the extra shift is measured and converted to radial velocity, v_r.

But what if the water is shallow, and we use the above equations to measure radial velocity? Further, how shallow does the water have to be so that the above equations start to incur significnat error? We can now answer that question with the results derived in the previous memo. Let us start by replacing Eq. (4) above by the shallow-water equivalent. This is done nearly by inspection to give Eq. (5) below:

$$S_D(\omega) = a_+ \exp(i\xi_+)\partial\left(\omega - \sqrt{\frac{g4\pi f_{MHz}}{300}\tanh\left(\frac{4\pi f_{MHz} d}{300}\right)} - \frac{4\pi v_r f_{MHz}}{300}\right) + a_- \exp(i\xi_-)\partial\left(\omega + \sqrt{\frac{g4\pi f_{MHz}}{300}\tanh\left(\frac{4\pi f_{MHz} d}{300}\right)} - \frac{4\pi v_r f_{MHz}}{300}\right)$$

Because the hyperbolic tangent is always less than unity, the Bragg peak positions are both moved inward toward DC from their deep-water positions. Therefore, if the deep-water positions are erroneously used as the reference, the error in Doppler frequency in Hertz is:

$$\varepsilon_{DF} = \frac{1}{2\pi}\left(\sqrt{\frac{g4\pi f_{MHz}}{300}} - \sqrt{\frac{g4\pi f_{MHz}}{300}\tanh\left(\frac{4\pi f_{MHz} d}{300}\right)}\right) \qquad (6)$$

Likewise, the error in radial velocity incurred by assuming deep water instead of shallow water is written by comparing Eq. (35) and Eq. (24) of our memo on waves, to give:

$$\varepsilon_V = \sqrt{\frac{g f_{MHz}}{4\pi \bullet 300}} - \sqrt{\frac{g f_{MHz} \tanh\left(\frac{4\pi f_{MHz} d}{300}\right)}{4\pi \bullet 300}} \qquad (7)$$

2. <u>Examination of Shallow-Water Velocity Error at SeaSonde Frequencies</u> Let us plot the error represented by Eq. (7) above. This is the error that would result if we tried to extract the current radial velocity, v_r, by subtracting the deep-water gravity wave Doppler when we should have subtracted the shallow-water Doppler for the Bragg waves. ***The plot also serves to show when depth matters for current mapping.***

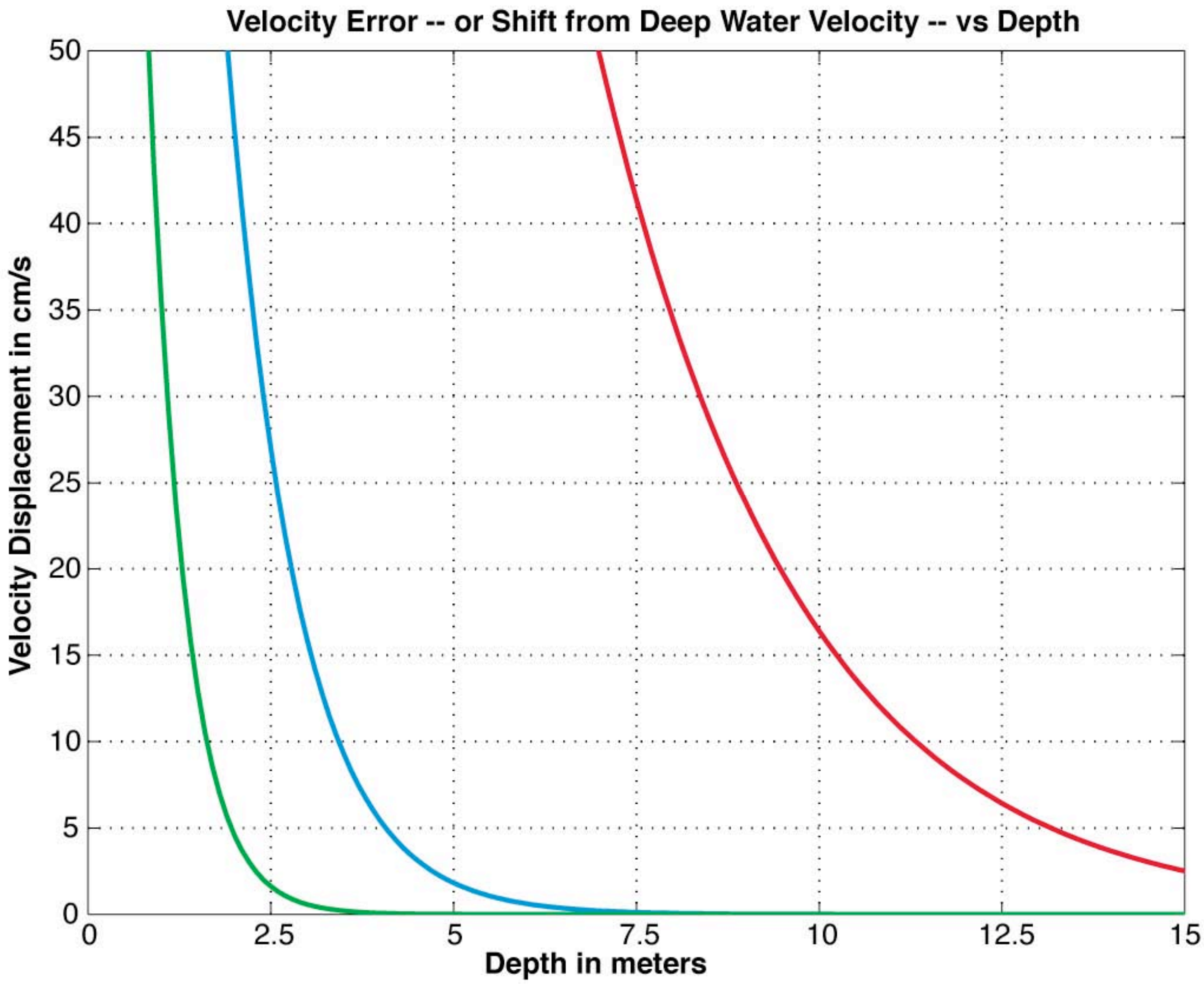

Figure 1. Radial velocity error (cm/s) incurred in using deep water dispersion relation when water is shallow, as a function of depth. Green curve is 25 MHz radar frequency; blue curve is 13 MHz radar frequency; red curve is 4.5 MHz radar frequency.

It is apparent that water must be quite shallow before it matters. For example, at 25 MHz radar operation, error is less than 5 cm/s as long as depth is greater than 2 m (6 feet). That's pretty shallow! At the lowest frequencies we operate, however, (4.5 MHz) depth will come into play quite often in the early range cells. The 5 cm/s error criterion demands depths exceeding 13.2 meters (41 feet). There are many regions of the U.S. where the depth may be less than this amount close to shore. For example, the East Coast (New Jersey), the West coast of Florida, and the Gulf of Mexico South off the Louisiana coast. We often see huge current vectors close to the radar at these locations in the first and/or second range cells. This results from the use of the deep-water uncorrected dispersion relation as part of our standard current extraction algorithms. On the West coast, however, (e.g., Oregon), water depth increases more rapidly with distance from shore as the continental slope is greater and the shelf is narrower. As a result, we rarely see the large spurious vectors close to the radar.

3. <u>Examination of the Shift in Bragg Peak Due to Shallow-Water</u> When the water is shallow, its effect is to shift the positions of the no-current Bragg peaks closer to DC, i.e., the zero-velocity or zero-Doppler center of the echo spectrum. A useful way to study this shift employs Eq. (6) above. However in this plot, we normalize the Doppler shift by dividing by the deep-water position (which would appear at unity). Thus, if the water is deep, the shift is zero and the Bragg peak occurs at unity, the upper edge of the plot. The lowest value of the y-axis represents DC, or zero-Doppler.

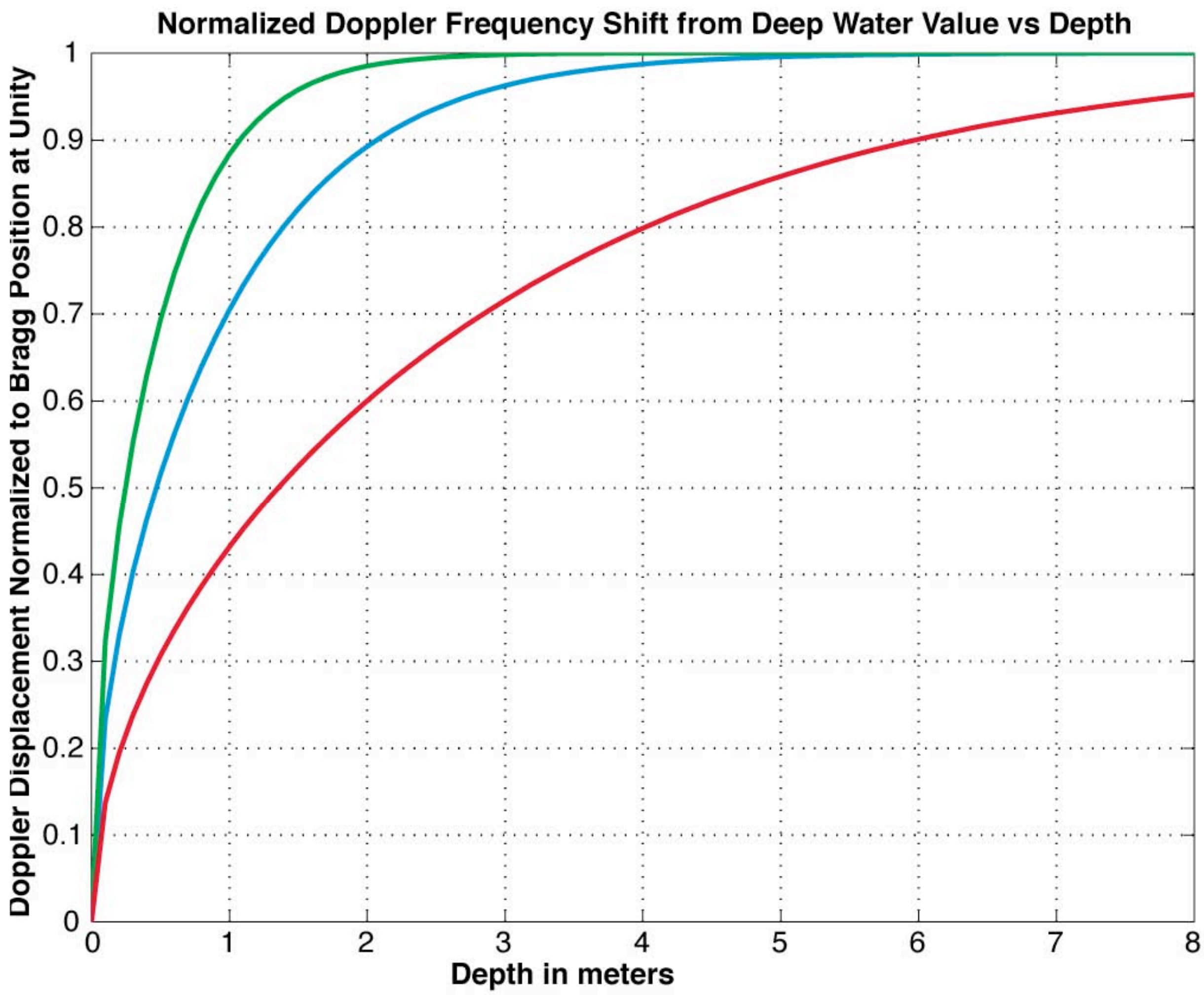

Figure 2. Normalized Doppler position of Bragg peak vs. depth. Upper y-value represents deep-water position; lowest y-value represents DC or zero-Doppler. Green curve is 25 MHz radar frequency; blue curve is 13 MHz radar frequency; red curve is 4.5 MHz radar frequency.

4. <u>Can We Correct in Software for Shallow Water</u>? It seems like it should be straightforward to correct for the shallow water in order not to incur error in radial velocity calculation. Or is it? After all, depth is well known from bathymetry charts; just measure it for each range cell and each bearing position near the radar, i.e., $d\left(r,\varphi\right)$, and use this "bathymetry map" along with Eq. (7) to remove the error. This correction might be applied even after 'Rads' or 'Radz' files had been created using the standard deep-water algorithms. Just correct the values of v_r at every point r,φ and write out a new, corrected 'Rads' or 'Radz' file.

Let's look at this further before undue euphoria. Wouldn't it be nice if, say, Range Cell #2 was shallow ***but constant depth*** for all bearings and across the span of the range cell? Unfortunately, this is rarely the case near shore, where the water is shallow. In fact, let's fix the bearing and just consider range variation. If we are in Range Cell #2 for a 4.5 MHz system, it typically spans 8 to 16 km; it is nominally centered on 12 km. If the range cell edges were sharp, the 6-km cell would span 9-15 km, but the Hamming window we must use extends the edges both ways by ±15%.

If you look at any bathymetry chart with a mean depth (along any bearing) centered at 12 km, ***it is very unlikely that the depth remains constant from 8-16 km.*** Near Rutgers off New Jersey, for example, my chart shows a depth spread along a bearing within 20° off the coast of perhaps 6 m at 8 km distance, varying out to 12 m at 16 km distance. If we enter the red curve on Fig. (1) above at 6 m depth, we get a velocity error that is off scale, perhaps as great as 75 cm/s; on the other hand, at 13 m depth, the velocity error drops to 5 cm/s. ***Therefore, depending on where you are on this bearing spoke the velocity error -- or shift from deep water -- varies over 70 cm/s, being in greatest error at the close edge of Range Cell #2.*** But on the other hand, first-order scatter from the Bragg waves originates all along the bearing spoke within this cell spanning 8-16 km simultaneously. Therefore, the Bragg peak -- in the absence of current -- would not be a narrow spike, but would be smeared over a significantly large Doppler spectral range that it would appear to arise from a current variation over 70 cm/s, although we are not even considering a current to be present yet. Since the span of currents we want to measure is usually much less than 70 cm/s, and we want an accuracy no worse than 5 cm/s, this smearing due to varying depth seems to render "correction in software" all but impossible! (Unless someone sees a solution somewhere that I am missing.)

5. <u>Onward toward Wave Measurement via Second Order in Shallow Water</u> When we use the second-order sea-echo spectral region, we are no longer scattering from Bragg waves half the radar wavelength. The philosophy that we had applied above is no longer relevant. The second-order Doppler continuum we see is due to the longest, highest waves present within the scattering area, as these interact with the shorter waves near the Bragg length. This allows us to measure the "sea state", i.e., the strength of the biggest stuff out there. The heights of the shorter Bragg waves even at 4.5 MHz is sufficiently small that it is of little or no interest to anyone (except to use as "tracers" for underlying currents as we do at first order).

So now the effects of shallow water take on another nature. When a long, high wave starts in deep water (its normal origin) and propagates into shallow water where the radar can observe it via second-order scatter, how does it change? Surfers know that only when long swells move into shallow water, do things become interesting. That's because the long wave steepens so a surfer can gain some traction. "Steepening" means two things: it becomes higher and shorter in length. When this happens, it can break, usually as a "spilling" breaker that we all love to see. When a wave breaks, its kinetic energy dissipates as heat, a nonlinear process. Up to that point, linearity and energy conservation can be used to describe how the wave changes height and length, as its temporal period remains constant.

Remember, in the last section of our previous memo, we made a point of the fact that a solitary wave -- once started -- always retains constant temporal frequency; all its other properties can be changed by shallow water, including its height, its length, and its direction. Yes, direction also. If a long wave comes into the coast from deep water at a narrow angle to the coastline, it refracts toward shore so we see it approaching the beach always perpendicular. Who has ever seen a long wave come in at an angle to the coast? Not possible if there is shallow water, as the latter causes it to refract in direction toward the shallow region, just like light through a prism.

Kinsman (1965) has nicely readable treatments of linear waves in deep and shallow water. To keep things simple here, we assume a 2-D geometry, i.e., just width and water depth, no direction, like we did in our prior memo on waves. If this long deep-water wave were observed via its second-order echo in deep water, we would see one type of Dopppler spectrum. However, if it steepens as it moves into shallow water, and the radar's range cells observe it in shallow water, we will see a differently shaped spectrum. Then, if we extract wave information for the latter situation using deep-water algorithms, we will get a wrong answer. How wrong? Where does it begin to matter? We can get an initial appreciation of this question by examining what happens to a deep-water wave of constant temporal frequency as it moves into shallow water, but for the 2-D geometry that keeps it simple.

6. Steepening as a Constant-Frequency Wave Moves into Shallow Water Let us repeat an equation from the preceding section for a 2-D deep-water wave given in terms of its temporal frequency:

$$\eta\left(x,t\right) \;=\; h_{\infty}\cos\left(\kappa_{\infty}x \;\pm\; \omega t \;-\; \xi\right) \qquad (8)$$

where the subscript ∞ stands for infinitely deep water, and $\kappa_{\infty} \equiv \dfrac{\omega^2}{g}$.

Kinsman [*Wind Waves,* Prentice-Hall, Inc., 1965, 676 p.] shows in Section 3.5 that as this wave propagates directly into shallow water with depth d, its transformation is given by his Eq. (3.5.13):

$$\eta\left(x,d,t\right) \;=\; h_{\infty}F\left(\kappa,d\right)\cos\left[\kappa\left(\kappa_{\infty},d\right)x \;\pm\; \omega t \;-\; \xi\right] \qquad (9)$$

The multiplicative factor $F\left(\kappa,d\right)$ begins at unity in deep water, and is modified by the depth d. The shallow-water spatial wavenumber $\kappa = \kappa\left(\kappa_{\infty},d\right)$ is found by solving the equation:

$$\kappa\tanh\left(\kappa d\right) = \kappa_{\infty} = \frac{\omega^2}{g} \qquad (10)$$

and the multiplicative -- or wave amplification -- factor is given by:

$$F(\kappa, d) = \frac{1}{\sqrt{\left[\frac{\kappa_\infty}{\kappa}\right]\left\{1 + \frac{2\kappa d}{\sinh(2\kappa d)}\right\}}} \qquad (11)$$

7. Timely Application to Tsunami Waves A tsunami – believe it or not – is just a surface gravity wave, like our examples above. It is just long as hell and carries a lot of energy! And its temporal period may be 100 times longer than the familiar sea-state waves above. These waves are generated as impulse responses to a very localized seismic (earthquake) point source. Such a source dumps massive amounts of energy into moving the ocean water around it.

These waves have temporal periods between 10 and 60 minutes, with 20 minutes being a typical value. If we use the "deep water" dispersion equation $\kappa_\infty = \frac{\omega^2}{g}$ to calculate the wavenumber and wavelength, we get a wavelength $L_\infty = \frac{2\pi}{\kappa_\infty} = \frac{gT^2}{2\pi} = 2250\ \text{km}$. If you think for a moment, the shallow-water-limiting case happens when the depth is less than about 10% of the spatial wavelength in deep water. This would require a depth greater than 225 km. There is nowhere on this planet where the ocean depth comes close to this amount. Therefore, ***this means that everywhere, a tsunami wave is in shallow water.*** In fact, the water is so shallow that all of hyperbolic functions can be replaced by their small-argument expansions, simplifying the equations greatly. Also, over the deep ocean basins, even an energetic tsunami wave has an amplitude of only ~10 cm. It's only when it moves into very shallow water that the wave steepens (the height increases and the spatial period decreases), causing the horrible damage at coastlines.

In this case, we can simplify Eq. (11) above and write an equation in place of (9) for the tsunami wave as follows (where x is directed away from shore and the wave travels toward shore):

$$\eta(x,t) = \frac{h_\infty}{\sqrt{2}\left(\frac{\omega^2 d}{g}\right)^{1/4}} \cos\left(\frac{\omega}{\sqrt{gd}}\,x + \omega t - \xi\right) \qquad (12)$$

The shallow-water wavenumber as seen above is $\kappa = \dfrac{\omega}{\sqrt{gd}}$, and hence the shallow-water wavelength, written in terms of the tsunami temporal period, is $L = T\sqrt{gd}$. For our 20-minute tsunami wave in 4000 meters water, this wavelength is 240 km; however, in near-shore waters that are 40 meters deep, the wavelength shortens to 24 km.

Using Eqs. (15) and (30) from our previous paper, we need to write an expression for the velocity potential -- Eq. (13) below:

$$\varphi\left(x, z, t\right) = \frac{-h_\infty g^{5/4}}{2^{1/2}\omega^{3/2}d^{1/4}} \sin\left(\frac{\omega}{\sqrt{gd}}x + \omega t - \xi\right)\cosh\left(\frac{\omega}{\sqrt{gd}}\left(z + d\right)\right)$$

Now, let us look at two important velocities for the tsunami. The first is the phase or group velocity of the crest/trough pattern of the tsunami wavetrain (assuming the water remains constant depth over its shallow-water wavelength). From our previous paper -- and by inspection of the above equations -- the phase and group velocities in the shallow-water limit become $c \Rightarrow \sqrt{gd}$. So, for example, start with (infinitely) deep water where $c = \dfrac{g}{\omega}$, the tsunami phase speed would be 1870 m/s or 6740 km/hr. That's fast! It's faster than the speed of sound in water (1500 m/s) and *much* faster than sound speed in air (300 m/s). However, we said earlier that there's nowhwere on earth that's "deep water" for our tsunami wave. But if we use the shallow-water formula, the phase/group speed in 4000-m deep water is ~200 m/s or 713 km/hr. Near shore where depth may drop to 40 meters, these numbers for phase speed drop by a factor of ten to 20 m/s or 71 km/hr.

Finally, we have Eq. (13) above for the velocity potential. Its spatial derivatives are the velocities of water particles -- ***not*** of features like the crest/trough. Thus, near the surface, the horizontal particle velocity -- by definition -- is $v_x = \dfrac{\partial\varphi\left(x, 0, t\right)}{\partial t}$.

Using this in Eq. 30, we get:

$$v_x = \frac{-h_\infty g^{3/4}}{2^{1/2}\omega^{1/2}d^{3/4}} \cos\left(\frac{\omega}{\sqrt{gd}}x + \omega t - \xi\right) \qquad (14)$$

Bragg waves seen by an HF radar are like water particles. So if these Bragg waves are being transported back and forth by the tsunami waves, this is effectively a current pattern. And so the particle velocity becomes the radar Doppler-producing velocity. So as one can see from the above equation, ***the higher the tsunami amplitude or height, the faster is the tsunami current visible to an HF radar.*** Since the amplitude in deep water, as we have discussed, is a meaningless quantity (the water is never infinitely deep for a tsunami), let's reference things to 4000 meters depth and call the tsunami height there h_{4000}. Then by inspecting Eq. (12) above, we can re-write this in terms of our 4000-m reference height as:

$$\eta\left(x,t\right) \;=\; h_{4000}\left(\frac{4000}{d}\right)^{1/4} \cos\left(\frac{\omega}{\sqrt{gd}}\,x \;+\; \omega t \;-\; \xi\right) \qquad (15)$$

Likewise, we can write our particle or Bragg current speed from Eq. (14) in terms of this more meaningful reference height of 4000 meters depth:

$$v_x \;=\; \frac{-\,g^{1/2}\,4000^{1/4}\,h_{4000}}{d^{3/4}}\cos\left(\frac{\omega}{\sqrt{gd}}\,x + \omega t - \xi\right) \qquad (16)$$

Let's put some numbers in the above equations to get a feel for the tsunami wave behavior in coastal waters. Let's assume the tsunami height at 4000 meters depth is only $h_{4000} = 10\,\mathrm{cm} = 0.1\,\mathrm{m}$. Then when the water is 40 meters deep, the tsunami amplitude to be used in Eq. (15) above grows to 31.6 cm. Its particle or current Bragg speed for use in Eq. (16) is 16 cm/s. But as we saw previously, the phase speed of the crests themselves at 40-m depth is 20 m/s, and their crest/crest wavelength is 24 km.

To finish up, let's see what happens at 4 m depth (about twice the height of a human). The tsunami height has now increased to 56 cm; the crest/crest wavelength is 7.6 km; the phase speed of the crests is 6.3 m/s; the Bragg/particle speed is 88 cm/s.

All of the above treatment is based on the assumption of linear wave physics. This means that wave energy is conserved as they move into shallow water. Surprisingly, this is a very good assumption, even when the water is quite shallow. Part of the reason for this is the enormous amount of energy in a tsunami wave -- it takes a lot of dissipation to get rid of it. Among phenomena that can rob energy from the wavetrain are bottom friction and viscosity (conversion to heat); wave breaking; and nonlinear wave-wave interactions (transferring energy into other waves, into water turbulence, and into currents). These effects happen, but are minimal until the wave gets very close to shore.

Nonlinearities finally win the battle at the shoreline. After all, one can ask, if our sample tsunami above has an amplitude of only ±56 cm in 4 meters depth, why am I reading about monster 20-foot waves doing all the damage? The rise of the tsunami wave and the mass of water involved must go somewhere. When it encounters obstacles (rocks, piers, breakwaters, seawalls, protrusions in only a few feet of water), these cause breaking and re-direction, causing the water to rise up in some regions, while calmer in others. Picture a seawall holding back the rising water until it reaches the top; like a breaking dam or dike, water rushes over it in a torrent. Or, the seawall may suddenly break in one region, causing water to roar through the opening into vulnerable areas normally thought protected by the seawall.

8. <u>Limits of Linear Wave Theory and Shallow Water</u> All of the results thus far -- including our consideration of tsunamis above -- has been based on linear wave theory, as presented in our previous memo titled 'Surface Waves.doc'. First of all, what does this mean?

The equations we employed based on physical principles (e.g., Navier-Stokes from Newton's second law) are nonlinear, in that the desired quantities (velocities and height) occur in some cases as squares or products of each other. By invoking small-argument perturbation assumptions, we can neglect the nonlinear product terms, and end up with the simpler linear differential equations we solved and used for waves. As these linear waves move into shallow water, remaining linear, simple provable laws apply to them. Energy in the wavefield is conserved, and depth changes cause a dispersion change as well as simple linear refraction (like light going through a prism), but no energy change or transformation to other forms or parts of the wave spectrum.

Perturbations involve a "smallness parameter". In this case, it is wavenumber times waveheight. Higher-order nonlinear terms and their effects -- which are neglected -- are proportional to this smallness parameter. Stokes' works (1847) and Kinsman's book (1965) examine the higher-order terms due to nonlinearities for solitary wavetrains on deep water. Barrick and Weber [On the Nonlinear Theory for Gravity Waves on the Ocean's Surface, *J. Phys. Oceanog.*, vol. 7, pp. 3-21, 1977] extended this treatment (to third order) for a continuous spectrum of waves on deep water (rather than a solitary wave). Second-order sea echo used for wave inversion with HF radar depends on these derivations by Barrick and Weber. Nonlinear waves in shallow water is a subject yet to be treated in the generality of multiple (spectral) wave representations. We have derived and proven that for HF wave inversion, an upper limit on deep water is definitely reached beyond which these second-order perturbation models break down and wave information can no longer be extracted. We have called this the saturation limit, where second-order is so strong that it is indistinguishable from first order. The relevant smallness parameter is radar wavenumber times rms waveheight. When this parameter exceeds 0.5, both linear (first order) and nonlinear (second-order) theory breaks down and wave extraction is no longer possible. This parameter is, in fact, the radius of convergence of the perturbation series. This means that even if one had the heart and endurance to calculate third order, n-th order, etc., none of it would work because the radius of convergence -- the smallness parameter --has exceeded unity.

Besides HF radar limits on second-order utility, what about tsunamis? These are very long, energetic waves in very shallow water. ***It is certainly a fair question to ask: when and where is any of the stuff of the preceding section valid? How does water***

depth enter into it? Is there a depth at a given distance from shore where one can no longer use the linear theory results given above?

To this end, one can demonstrate this with a simple Stokes or solitary wavetrain. After all, a tsunami comes close to this behavior, being in fact just one or two single waves. The first two terms of a Stokes wave on deep water are:

$$\eta\left(x,t\right) = a\cos\left[\kappa x - \omega t\right] + \frac{\kappa}{2}a^2\cos\left[2\left(\kappa x - \omega t\right)\right] + \ldots \quad (17)$$

The second term on the right is the second-order nonlinear contribution. It is a bound second harmonic of the linear sinusoidal first-order wave, meaning it does not follow the deep water dispersion relation but stays attached to the fundamental (first term), traveling at its phase velocity. The perturbation "smallness parameter" and radius of convergence is $\kappa a/2$, where κa is the wave slope. When this is sufficiently steep that the second term becomes comparable in magnitude to the first, then the entire series becomes meaningless and nonlinearities (not predictable with the present theory) dominate. Also note that the second term begins to make the sinusoidal profile distort toward a trochoid, i.e., flat at the troughs and pointy at the crests, like the steep ocean waves we see pictures of.

In terms of the perturbation expansion used by Barrick and Weber (1977), the factor κ before the second term is the "hydrodynamic coupling coefficient" when the roughness becomes a solitary wave. This is the hydrodynamic coupling coefficient for a deep-water gravity wave. What is the coupling coefficient for a shallow-water gravity wave in the limit where small-argument terms can be used for the hyperbolic functions? We deal with this question next, as it begs an answer for tsunami waves.

The general second-order hydrodynamic coupling coefficient for shallow-water gravity wave was derived and published in Barrick and Lipa [The Second-Order Shallow-Water Hydrodynamic Coupling Coefficient in Interpretation of HF Radar Sea Echo, *IEEE J. Oceanic Engr.*, vol. OE-11, pp. 310-315, 1986]. One needs to simplify Eq. (16) of that document in the small-depth limit. When this is done, in place of the coupling coefficient, κ, above, one obtains $3/2d$ (where d is the water depth).

For a (the amplitude of the linear sinusoidal wave), we should use the factor multiplying the cosine term in Eq. (15) above for shallow water. Then in place of Eq. (17) above for deep water, we arrive at the following expression for the linear and next nonlinear second-order contribution for shallow water:

$$\eta\left(x,d,t\right) = h_{4000}\left(\frac{4000}{d}\right)^{1/4}\cos\left(\frac{\omega}{\sqrt{gd}}x + \omega t - \xi\right)$$
$$+ \quad \frac{3}{4}\frac{4000^{1/4}}{d^{5/4}}h^2_{4000}\cos\left[2\left(\frac{\omega}{\sqrt{gd}}x + \omega t - \xi\right)\right] \qquad (18)$$

Now we can address the question: ***at what depth do nonlinearities dominate to the point that linear wave theory (the first term) is no longer adequate?*** A simple criterion is this: when the magnitude of the second (nonlinear) term amplitude becomes one-half of the first (linear term). Call this threshold depth; it is given by:

$$d_{thresh} = \left(\frac{3}{2}h_{4000}\right)^{4/5} 4000^{1/5} \qquad (19)$$

Let us substitute some numbers into this from the Dec. 26, 2004 Sumatra tsunami. In that case, the amplitude in deep water (take to be 4000 m) was reported as 50 cm. Using this in Eq. (19), we get 4.2 meters for the depth for the onset of nonlinearity. On the other hand, if the amplitude in deep water is 10 cm, then the nonlinear onset depth is 1.15 m.

This all seemed surprisingly shallow to me at first, but I cannot find anything wrong with it. After all, the tsunami wave is very long and very low in amplitude in deep water. Nonlinearity depends on the amplitude and/or steepness, and this doesn't begin to become significant until one is very close to shore. There were reports, for example, of divers offshore in water 40-50 meters off India and Sri Lanka, who did not even know the tsunami wave had passed by until they saw the ravaged coastline or heard on their radios.

9. Plot of Wave Amplification Factor vs Depth One must admit, it is difficult to look at Eq. (11) above and tell what happens to the height of the wave as it moves into shallow water and d decreases. We want to examine this for different wave periods (or temporal frequencies), because wave period and temporal radian frequency are related by $T = 2\pi / \omega$. We will pick three wave periods typical of moderately long, high "sea-state" waves: 6 seconds; 10 seconds; and 14 seconds.

First, we substitute these periods and radian frequencies into Eq. (10) to solve for shallow-water wavenumber, and then substitute this into Eq. (11) to get our factor. This is shown in Fig. (3) below for the three different wave periods as a function of water depth.

The green curve is the shortest-period wave: 6 seconds. Blue is an intermediate period wave: 10 seconds. And the red is the longest wave period: 14 seconds. As expected, the long-period waves feel the bottom first as the depth decreases. Starting at unity for very deep water, the effect is first of all, to decrease the height a little, i.e., about 8%. Then as water gets shallower, the height begins to increase rapidly, heading toward infinity as the shallowest water is encountered. However, the height never gets too large, because the wave will break, bringing its height back down to a sustainable level. As it increases but before it breaks constitutes the surfers' dream of catching the "perfect wave".

10. <u>Plot of Wave Spatial Period vs Depth</u> The wave's spatial period must be examined as well as its height, as it propagates into shallow water. As its height increases, its spatial period decreases, according to Eq. (10) above. The two together make the wave steepen, i.e., its slopes increase precipitously. The spatial wave periods for deep and shallow water are obtained from the wavenumbers very simply as:

$L_{\infty} = 2\pi / \kappa_{\infty}$ for the deep-water wave, and $L = 2\pi / \kappa$ for the shallow-water wave at depth d. These are plotted in Fig. (4) below at the three temporal wave periods.

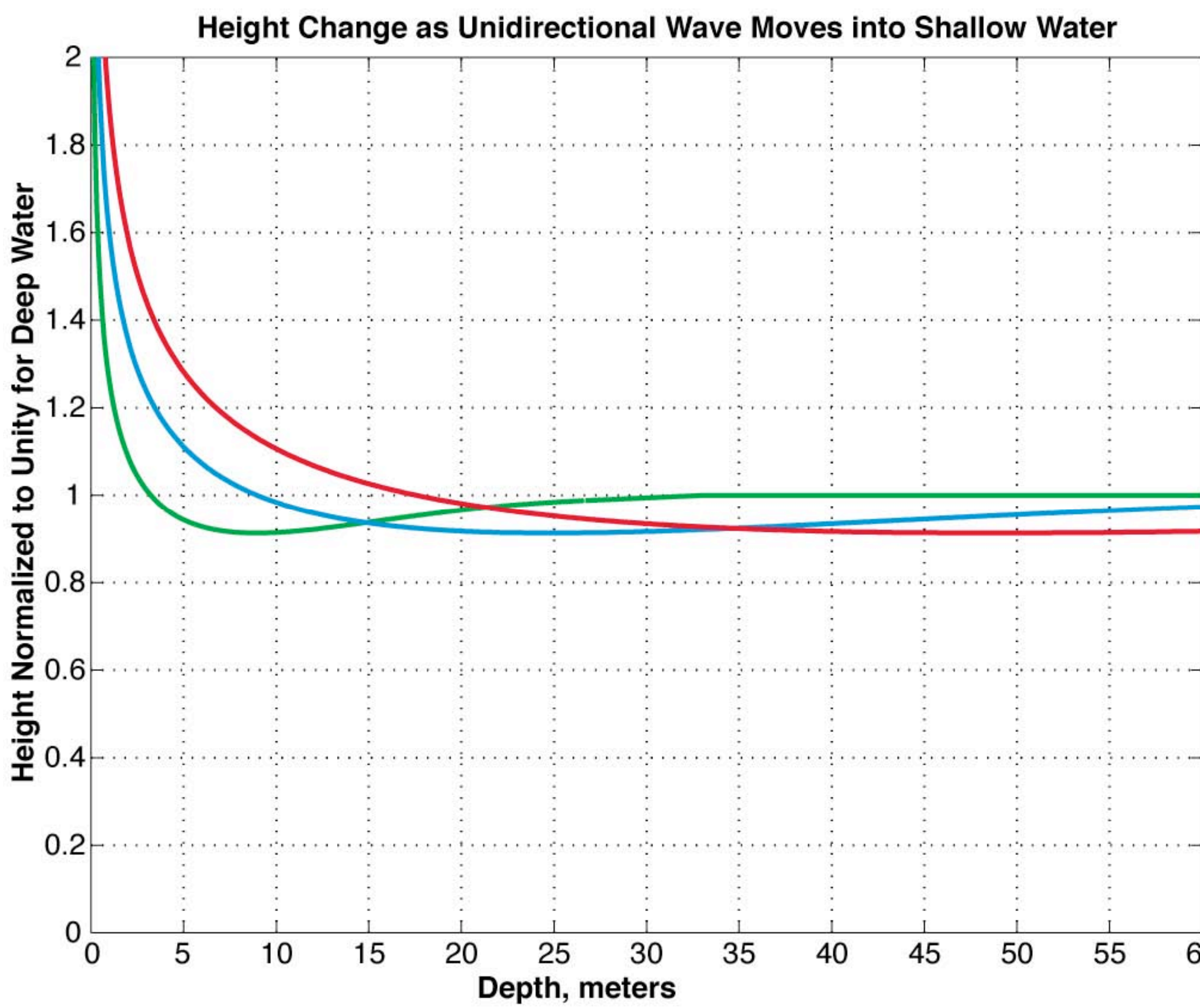

Figure 3. Normalized wave amplification factor as it propagates into shallow water, vs. water depth. Three wave periods are considered: green curve is 6 seconds period; blue curve is 10 seconds period; red curve is 14 seconds period. Waveheight first decreases by about 8%, then increases rapidly as water depth decreases to left.

In the case of wave height, we see that the waves must move into quite shallow water before the amplitude really begins to take off. Look at the situation in Fig. (3) above at 10 meters depth. None of the wave amplitudes have changed more than 10% yet from their deep-water values. That's pretty shallow, actually, because a 14-second swell can be nearly 8 meters (25 feet) high in deep water. However, although the height has not increased that much, the wave period has changed quite a bit for the 14-second wave. From Fig. (4) it has decreased from 310 meters to135 meters at the 10-meter isobath. This, in turn, makes the wave quite steep. For the 6-second wave, however, the water depth must reach 2 meters before the same amplification and steepness conditions are realized. This happens because spatial dimensions go as the square of the temporal frequencies, as seen from all the equations above.

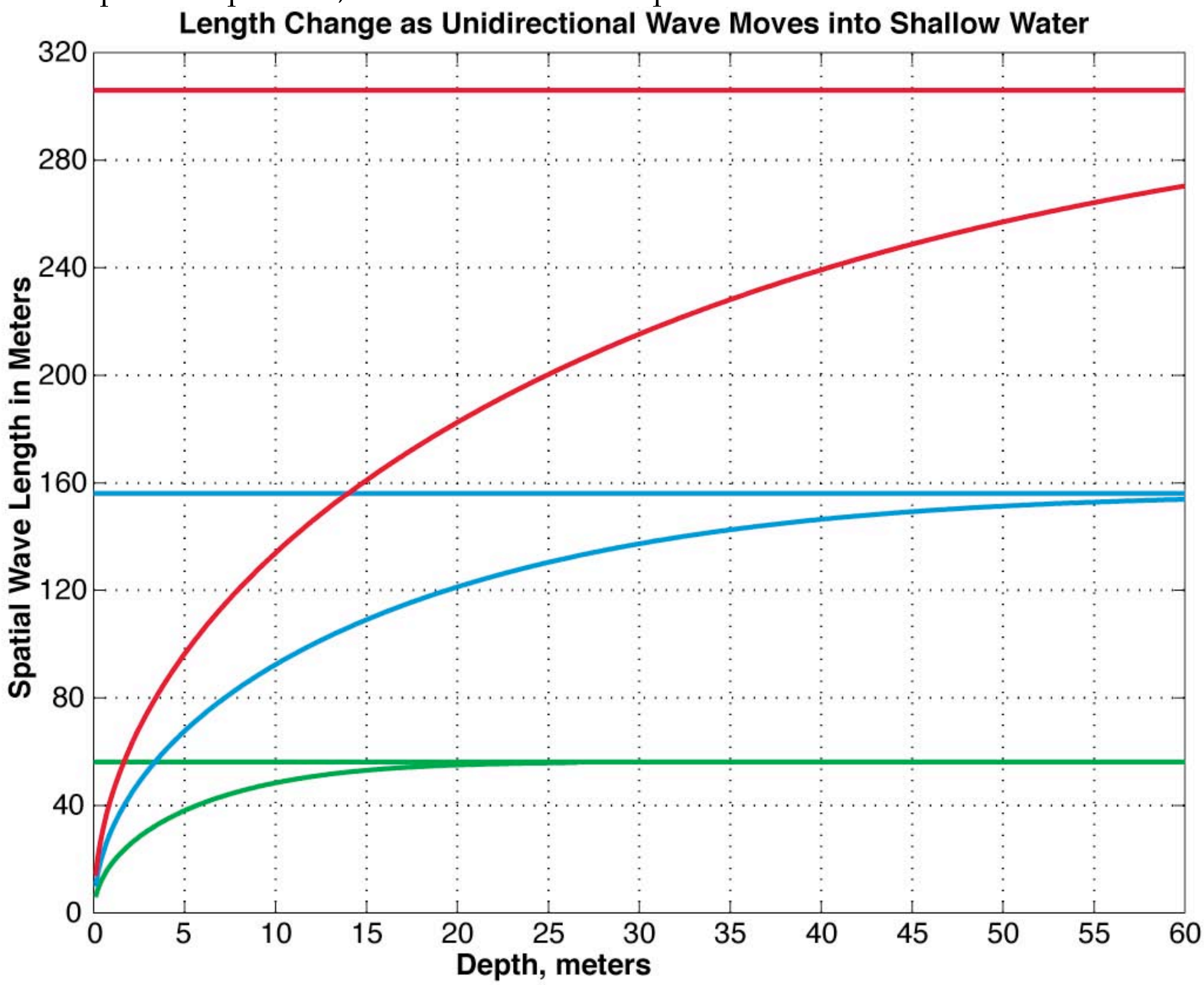

Figure 4. Wavelength or spatial period change as wave moves from deep water (right) to shallow water (left), vs. water depth. The flat, straight horizontal lines are the asymptotic deep-water wave periods with which the curves merge. Three wave periods are considered: green curve is 6 seconds period; blue curve is 10 seconds period; red curve is 14 seconds period. Waveheight first decreases by about 8%, then increases rapidly as water depth decreases to left.

11. <u>Effect on Second-Order HF Doppler Spectrum</u> We saw above that period changes most rapidly with wave frequency, while height changes more slowly and only for very shallow water. Height-squared is more representative of wave energy (along with the speed and volume transport), and energy is conserved until breaking or some other mechanism converts its kinetic energy into some other form.

First of all, the contrasts between first and second-order HF echo spectra regions are worth summarizing. First order peaks -- used for current mapping -- come from short waves that are said to be in equilibrium with the wind, i.e., they don't change in amplitude much as the longer, higher "sea-state" waves increase. On the other hand, the second-order peaks readily respond to the heights of the longer waves. With practice, you should be able to just look at them and -- like Belinda and I -- come within a few percent of estimating the waveheight out there. If the waveheight doubles, the wave energy quadruples, and the second-order peak amplitudes should also quadruple (increase by 6 dB) with respect to the Bragg peak amplitudes. These second-order peaks are quite sensitive to waveheight -- up to the point where we say the perturbation theory breaks down and second-order inversion methods fail. This limit is described in our website literature.

Since we saw that the height and energy of long, high sea-state waves *do not change* until quite shallow water is encountered, we might expect that the second-order HF radar echo spectrum would be somewhat weakly dependent on depth, until we get into really shallow water. This is indeed the case. We repeat in Fig. (5) a curve calculated in Lipa and Barrick [Extraction of sea state from HF radar sea echo: Mathematical theory and modeling, *Radio Science,* vol. 21, pp. 81-100, 1986] for the second-order echo spectrum of deep water waves as the propagate into shallow water. In addition to the wave steepening discussed above, there are other factors that influence the scattering process. The most significant is the coupling coefficient that comes from hydrodynamic and electromagnetic theory solved by perturbation methods to second order. These expressions are all given in the above reference and were used in calculating the Doppler spectrum. Read some of our 1980s works to get a flavor for this subject.

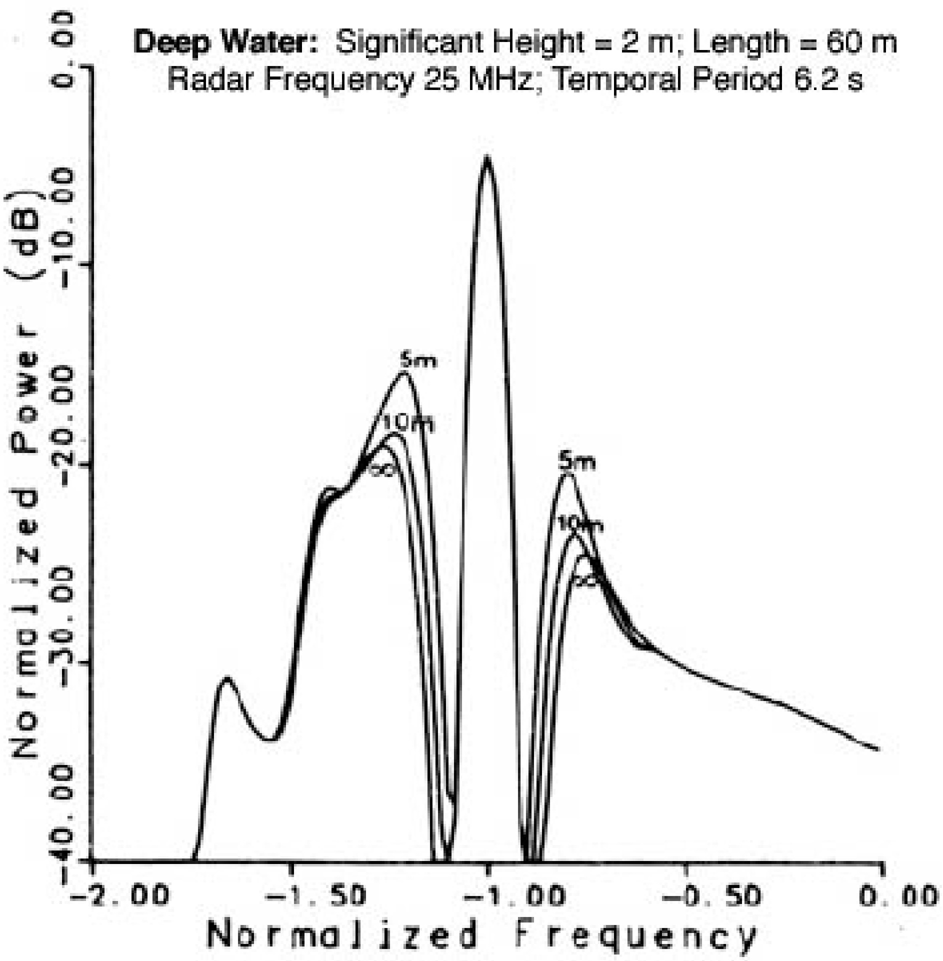

Figure 5. First and second-order Doppler spectra for waves moving offshore at 45°. We show only the dominant negative spectrum side, the highest peak being the first-order Bragg echo. These were calculated for different water depths at 25 MHz operation. The wave parameters are given at the top of the figure for deep water.

In this theoretical spectrum, only the negative Doppler side of DC is shown, nomalized so the first-order Bragg peak appears at unity Doppler frequency. Waves are moving offshore for a straight coastline at an angle of 45°, so the negative Doppler spectral side indeed dominates for this model case. A wave spectrum represented by a Pierson-Moskowitz model was used in these calculations, with a broad angular distribution about the maximum wave direction (a favored form for a wave spectrum). Its parameters are such that in deep water for 25 MHz radar operation, the significant waveheight is 2 m, the dominant wave spatial period or wavelength was 60 meters, which corresponds to a temporal period of 6.2 seconds.

Resulting second-order Doppler spectra curves are shown for three depths that are uniform across the scattering region. Infinitely deep water is represented by ∞, and depths of 5 meters and 10 meters. One can see that going from infinitely deep to 10 meters depth changes the amplitude of the Doppler echo by only a dB. From there to 5 meters depth, the change is more significant, about a 4 dB increase. This slow increase is consistent with the actual slow change in waveheight as depth decreases, as discussed in preceding sections.

This curve -- although done for 25 MHz radar operation -- is easily scaled to other radar frequencies. Things always scale as the radar wavelength. So for example, the same normalized curves would apply for:

a) Radar freqeuency operation at 12.5 MHz; significant waveheight of 4 meters; dominant ocean wavelength of 120 meters; and depths of ∞, 10 meters, 20 meters.

b) Radar freqeuency operation at 5 MHz; significant waveheight of 10 meters; dominant ocean wavelength of 300 meters; and depths of ∞, 25 meters, 50 meters.

Two conclusions can be drawn from the above curves for what errors are incurred if you neglect shallow water (when it is important), using only deep-water wave extraction algorithms: (1) You will overestimate waveheight; a 3 dB increase in second-order-echo energy due to shallow water will give too high an estimate of waveheight by roughly 40% (a factor of square root of two); (ii) You will overestimate the temporal wave period; this is seen as the peaks of the second-order echo move closer to the Bragg peak position, implying a lower wave temporal peak frequency.