Reflection of Electromagnetic Waves from Slightly Rough Surfaces

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I. *Introduction*

The problem of reflection of waves from a rough or corrugated surface is of interest in a number of different fields. In particular, the problem occurs in the propagation of radio waves over rough ground or over the sea. The general problem of reflection from rough surfaces appears to be difficult. However, a number of investigators have been able to make progress by dealing with special cases or by making suitable assumptions. For example, the radio problem in which the impressed field originates at a point has been studied by E. Feinberg [1]. L. V. Blake [2] has applied probability theory to the problem of calculating the reflection of radio waves from a rough sea. The reflection from the very rough surface formed by the edges of an infinite set of plates has been investigated by J. F. Carlson and A. E. Heins [3]. Also, in addition to the studies of W. S. Ament[1] mentioned below, I understand that a considerable amount of work on this subject, which is unpublished as yet, has been done by Mr. Twersky [7] of New York University, and by Messrs. Norton and Hufford and their associates at the National Bureau of Standards.

Here we shall be concerned with the reflection of plane electromagnetic waves from a surface $z = f(x, y)$ which is almost, but not quite, flat. The small deviations of this surface from the x,y-plane are of a random nature. Except in Section 7, the surface is assumed to be a perfect conductor. Although in practical cases the surfaces are usually much rougher than is assumed here, our problem has the virtue of being one of the simplest which still shows the effect of roughness.

The roughness of the surface is described by a "roughness spectrum" o "roughness distribution function" $W(p, q)$. When the surface is expressed as

Paper presented at the June, 1950, Symposium on the Theory of Electromagnetic Waves, unde the sponsorship of the Washington Square College of Arts and Science and the Institute fo Mathematics and Mechanics of New York University and the Geophysical Research Directorate of the Air Force Cambridge Research Laboratories.

[1] I am indebted to Mr. Ament of the Naval Research Laboratory for an opportunity t study some of his work before its publication. I also wish to acknowledge the help I have received from discussions of the general problem of reflection with K. Bullington of the Bell Telephone Laboratories.

the sum of two-dimensional Fourier components, $W\ p\ \ q\ \ dp\ dq$ represents the relative strength measured by th ir contribution to the m an square valu of $f\ x, y$ of tho components which go through between p and $p + dp$ radians meter in the x dire tion and through between q and $q + dq$ in the y direction. If the average d ance betw n th hills on the surface is larg , and the surface is smooth ex ep for the hill , $W\ p, q$ will be appreciably different from zero only for small values of p and q. The associated auto-correlation function of the surface, whi h may be expressed as the Fourier transform of $W\ p, q$, is not used here although it do oc ur in the work of W. S. Ament.

The reflected field is d termined by a method similar to that used by Rayleigh [4] to study the reflection of acoustic waves from rough walls. The expressions which we obtain for the field are not exact since the boundary conditions at the surface are satisfied only to within $O(f^2(x, y))$, i.e. to within terms of the second order—a shortcoming forced upon us by the increasing complexity of our successive approximations. The two cases corresponding to horizontal polarization (incid nt E vector parallel to x,y-plane) and vertical polarization (incident H vector parallel to x,y-plane , respectively, are considered.

After expressions for the components of the field are obtained, various averages are computed, the average being taken over many surfaces which are different but which have the same statistical properties. In parti ular, the average value of the reflected field leads to an expression for the reflection coefficient. It is found that this reflection coefficient depends upon the polarization in somewhat the same way as does the reflection coefficient for an almost, but not quite, perfectly conducting plane. Also, when the average distance between hills is large, the reflection coefficients for both the horizontal and vertical polarizations reduce to the same expression. By a method similar to the one used in the study of Fraunhofer diffractions, W. S. Ament has obtained an expression for the average reflection coefficient when the distance between hills is large and they are such that they do not cast any shadows (with respect to the incident wave). Our approximate expression agrees with the first two terms in the expansion of Ament's expression, which is as much of an agreement as the accuracy of our work allows.

Closely associated with the problem of reflection is the problem of surface wave propagation. This corresponds loosely to the case of grazing incidence and vertical polarization; a modified form of the reflection analysis may be used to obtain an expression for the propagation constant of the surface wave. It is found that, roughly speaking, the Fourier components of the surface whose wavelengths are much greater than that of the electromagnetic wave tend to produce attenuation through scattering, while the guiding action of the surface is due to the components of shorter wavelength. This is in accord with the results of earlier studies of surface waves on corrugated surfaces [5, 6].

The method used to study reflection from a slightly rough but perfectly conducting surface may be extended to take into account the electrical properties of the reflecting medium. This is done in Section 7 for the case of horizontal polarization, the magnetic permeabilities of the two media being assumed equal. There are two reasons for this study. The first is to determine the additional amount of complication introduced. The second is to show that an annoying difficulty encountered in the perfect conductor case, namely that the integral for the mean square value of E_z (i.e. the component of electric intensity which is approximately normal to the surface) sometimes diverges logarithmically, may be removed by taking into account the finite conductivity of the reflector.

2. *Description of Rough Surface*

We shall take the equation of the perfectly conducting rough surface to be

$$z = f(x, y) = \sum_{mn} P(m, n) \exp\{-ia(mx + ny)\} \tag{2.1}$$

$$a = 2\pi/L$$

where the double summation extends from $-\infty$ to $+\infty$ for both m and n. The definition of a shows that $f(x, y)$ is periodic in both x and y with period L (assumed to be large). In order to make $f(x, y)$ real we impose the condition

$$P(-m, -n) = P^*(m, n) \tag{2.2}$$

where the asterisk denotes the conjugate complex quantity.

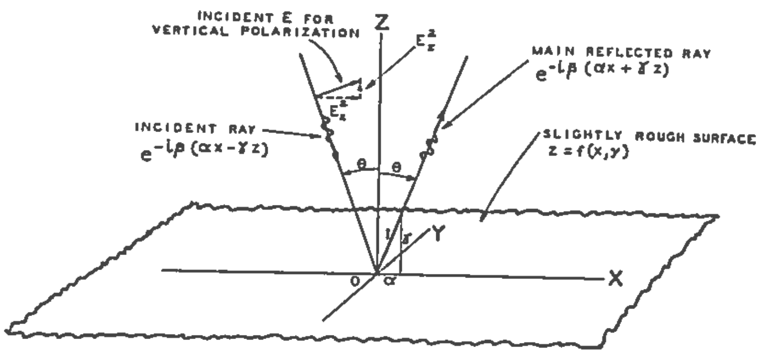

FIG. 1. Diagram showing the coordinate system and the incident E vector for vertical polarization. For horizontal polarization the incident E vector is parallel to the y axis.

The random character of roughness is introduced by taking the coefficients $P(m, n)$ to be independent random variables, subject only to (2.2). For the sake of being definite, we assume $P(0, 0)$ and the real and imaginary parts of $P(0, 1)$, $P(1, 0)$, $P(2, 0)$, $P(1, 1)$, $P(0, 2)$, $P(1, -1)$, etc. to be independent random variables distributed normally about zero. We assume further that, for assigned values of m and n, the four independent random variables formed by the real and imaginary parts of $P(m, n)$ and $P(m, -n)$ all have the same vari-

ance, i.e., the same mean square value. When we use angular brackets to denote average values, our assumptions tell us that

$$\langle P(m, n)\rangle = 0$$

$$\langle P(m, n)P(u, v)\rangle = 0, \qquad (u, v) \neq (-m, -n)$$

$$\langle P(m, n)P^*(m, n)\rangle = \langle P(m, n)P(-m, -n)\rangle = \pi^2 W(p, q)/L^2 \tag{2.3}$$

$$W(p, q) = W(|p|, |q|)$$

$$p = am = 2\pi m/L, \qquad q = an = 2\pi n/L$$

Here $\langle\ \rangle$ denotes that m and n are to be held fixed and the average taken over the universes of the real and imaginary parts of the $P(m, n)$'s. $W(p, q)$ is the roughness spectrum mentioned in the introduction, and p and q are radian wave numbers. Note that $\langle P^2(m, n)\rangle$ is zero, except when $m = n = 0$, by virtue of the real and imaginary parts of $P(m, n)$ having the same variance. Incidentally, the statistical properties of $P(m, n)$ were obtained by expressing the typical Fourier series term

$$\begin{aligned}(a_{mn} \cos amx + b_{mn} \sin amx) \cos any \\ + (c_{mn} \cos amx + d_{mn} \sin amx) \sin any, \qquad m > 0, n > 0\end{aligned} \tag{2.4}$$

as the sum of four exponential terms. This leads to four relations of the form

$$P(m, n) = \frac{a_{mn} + ib_{mn} + ic_{mn} - d_{mn}}{4}$$

and the properties of $P(m, n)$ follow when a_{mn}, $\cdots$, d_{mn} are assumed to be independent random variabl distributed normall about zero with the same variance, namely $4\pi^2 W(p, q)\ L$. The $4\pi^2$ arises from the fact that we have elected to measure p and q in radians meter instead of cycles meter.

Equation (2.1) defines a surface for each set of coefficients. As an example of the use we shall make of $W(p, q)$ we compute the average value of $f^2(x, y)$ as we go from surface to surface, holding x and y fixed all the while.

$$\begin{aligned}\langle f^2(x, y)\rangle &= \sum_{mnuv} \langle P(m, n)P(u, v)\rangle \exp\{-iax(m + u) \quad iay(n + v)\} \\ &= \sum_{mn} \langle P(m, n)P(-m, -n)\rangle \\ &\to \int_{-\infty}^{\infty} dm \int_{-\infty}^{\infty} dn\, \pi^2 W(p, q)/L^2 \\ &= \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \frac{W(p, q)}{4} = \int_0^{\infty} dp \int_0^{\infty} dq\, W(p, q).\end{aligned} \tag{2.5}$$

Here we have used (2.3). In going from the summation to the double integral we have let the period L approach infinity. It is seen that $W(p, q)\,dp\,dq/4$ represents the contribution to $\langle f^2(x, y)\rangle$ of those components in (2.1) having between p and $p + dp$ radians/meter in the x direction and between q and $q + dq$ radians/meter in the y direction.

3. *Incident Wave Horizontally Polarized*

We assume the components of the total electric intensity for $z > f(x, y)$ to be

$$E_x = \sum A_{mn}E(m, n, z)$$

$$(3.1)\qquad E_y = 2i \sin \beta\gamma z \exp\{-ia\nu x\} + \sum B_{mn}E(m, n, z)$$

$$E_z = \sum C_{mn}E(m, n, z)$$

where the summations extend from $-\infty$ to $+\infty$ for m and n and

$$(3.2)\qquad E(m, n, z) = \exp\{-ia(mx + ny) - ib(m, n)z\}.$$

The time factor $\exp\{i\omega t\}$ is understood. $b(m, n)$ is either a positive real or a negative imaginary number:

$$(3.3)\qquad b(m, n) = \begin{cases} [\beta^2 - a^2m^2 - a^2n^2]^{1/2}, & m^2 + n^2 < \beta^2/a^2 \\ -i[a^2m^2 + a^2n^2 - \beta^2]^{1/2}, & m^2 + n^2 > \beta^2/a^2 \end{cases}$$

where $\beta = 2\pi/\lambda$, λ being the wavelength of the incident wave. A_{mn}, B_{mn}, C_{mn} are constants which we shall determine approximately on the assumption that βf and the partial derivatives f_x and f_y are small (here and in what follows we shall often denote $f(x, y)$ by f) compared to unity.

The field obtained from (3.1) when the summations are omitted is the one which would occur if the perfectly conducting surface were flat (f 0). In (3.1) we take ν to be an integer so that the field is periodic in x and y of period L by virtue of $a = 2\pi/L$. It follows that the angle θ between the incoming ray and the z-axis is restricted to certain discrete values given by

$$(3.4)\qquad \begin{aligned} a\nu = 2\pi\nu/L = \beta \sin\theta = \beta\alpha \qquad a\nu \quad \beta \\ \alpha = \sin\theta, \qquad \gamma = \cos\theta \qquad 0 < \gamma. \end{aligned}$$

Since L becomes very large we can pick an integer ν which will correspond approximately to any angle of incidence. The leading term in E_y may be written as

$$\exp\{-i\beta(\alpha x - \gamma z)\} - \exp\{-i\beta(\alpha x + \gamma z)\}$$

where the first term represents the incoming wave and the second term the

main part of the reflected wave. It is seen that the direction cosines of the incident and reflected rays are $(\alpha, 0, -\gamma)$ and $(\alpha, 0, \gamma)$, respectively. From the definition of $b(m, n)$ it follows that

$$(3.5)\qquad b(\nu, 0) = (\beta^2 - a^2\nu^2)^{1/2} = \beta\gamma.$$

The exponential form (3.2) of $E(m, n, z)$ ensures that all three components of the electric intensity (3.1) satisfy the wave equation. The coefficients are determined by the relation div $E = 0$, which gives

$$(3.6)\qquad amA_{mn} + anB_{mn} + b(m, n)C_{mn} = 0,$$

together with the condition that the tangential com onent of E must vanish at the perfectly conducting surface $z = f$. If N denotes the unit vector normal to the surface, $N(E \cdot N)$ is the component of the electric intensity normal to the surface. The remaining portion of E, the tangential component, is $E - N(E \cdot N)$, all three components of which must vanish. Equating the x and y components to zero gives

$$(3.7)\qquad \begin{aligned} E_x - N_x(E_xN_x + E_yN_y + E_zN_z) = 0 \\ E_y - N_y(E_xN_x + E_yN_y + E_zN_z) = 0 \end{aligned}$$

If these two equations are satisfied the z component is also zero (if $N_z \neq 0$) as may be seen by multiplying the first by N_x, the second by N_y, and adding. The components of N are

$$(3.8)\qquad N_x = -f_xN_z, \qquad N_y = -f_yN_z, \qquad N_z = (1 + f_x^2 + f_y^2)^{1\ 2}.$$

We now assume βf, f_x, f_y all to be of the same order of smallness which, for the sake of simplicity, we shall denote by $O(f)$ instead of $O(\beta f)$. Likewise instead of $O(\beta^2 f^2)$ we shall write $O(f^2)$, and so on. In our work we shall neglect $O(f^3)$ terms and it will not be necessary to go beyond the leading terms in

$$(3.9)\qquad N_x = -f_x + O\ f^3), \qquad N_y \quad -f_y + O(f^3), \qquad N_z \quad 1 + O(f^2)$$

Near the surface $z = f$, i.e. near z 0, the leading term in E_y as given by (3.1) is $O(f)$, and we assume for the moment that E_x and E_z are also of this order. Then, neglecting $O(f^3)$ terms in (3.7), we obtain the two boundary conditions

$$(3.10)\qquad \begin{aligned} E \qquad N\ E_z = 0 \\ E_y \qquad N\ E_z = 0 \end{aligned}$$

which must hold at z f. Thus, if E_z is $O(f)$, then both E_x and E_y must be $O(f^2)$ at the surface. Of course this holds only for horizontal polarization. For vertical polarization it turns out that E_z is O 1) and both E_x and E_y are $O(f)$.

The problem now is to choose the coefficients in (3.1) so that the divergence relation (3.6) and the boundary conditions (3.10) are satisfied to within $O(f^2)$.

Writing

$$\sin \beta\gamma f = \beta\gamma f + O(f^3)$$

(3.11) $$E(m, n, f) = [1 - ib(m, n)f + \cdots]E(m, n, 0)$$

$$A_{mn} = A^{(1)}_{mn} + A^{(2)}_{mn} + \cdots$$

where $A^{(1)}_{mn}$ is $O(f)$, $A^{(2)}_{mn}$ is $O(f^2)$, etc., and expressing B_{mn}, C_{mn} in a similar way enables us to write the boundary conditions (3.10) as

(3.12) $$\begin{gathered} \sum [A^{(1)}_{mn} + A^{(2)}_{mn} + f_x C^{(1)}_{mn}][1 - ib(m, n)f]E(m, n, 0) = 0 \\ 2i \exp\{-ia\nu x\}\cdot\beta\gamma f \\ + \sum [B^{(1)}_{mn} + B^{(2)}_{mn} + f_y C^{(1)}_{mn}][1 - ib(m, n)f]E(m, n, 0) = 0 \end{gathered}$$

where we have neglected $O(f^3)$ terms. In this work we shall overlook questions of convergence although they may perhaps be treated by placing suitable restrictions on the components $P(m, n)$ of the surface f.

Equating the first order terms in (3.12) to zero gives

(3.13) $$\begin{gathered} \sum A^{(1)}_{mn} E(m, n, 0) = 0 \\ 2i \exp\{-ia\nu x\}\beta\gamma f + \sum B^{(1)}_{mn} E(m, n, 0) = 0. \end{gathered}$$

Likewise, the second order terms yield

(3.14) $$\begin{gathered} \sum [A^{(2)}_{mn} + f_x C^{(1)}_{mn} - ib(m, n)fA^{(1)}_{mn}]E(m, n, 0) = 0 \\ \sum [B^{(2)}_{mn} + f_y C^{(1)}_{mn} - ib(m, n)fB^{(1)}_{mn}]E(m, n, 0) = 0. \end{gathered}$$

As (3.2) shows, $E(m, n, 0)$ is the exponential function of x and y which occurs in a double Fourier series. Hence the first of equations (3.13) requires $A^{(1)}_{mn} = 0$. In order to interpret the remaining equations we need the following results. Writing u, v for m, n in (2.1) and using the definition of $E(m, n, z)$ leads to

(3.15) $$\begin{bmatrix} f \\ f_x \\ f_y \end{bmatrix} = \sum_{uv} \begin{bmatrix} 1 \\ -iau \\ -iav \end{bmatrix} P(u, v)E(u, v, 0)$$

whence, upon setting $m = u + \nu$, $n = v$,

(3.16) $$\begin{aligned} \exp\{-ia\nu x\} f &= \sum_{uv} P(u, v)E(u + \nu, v, 0) \\ &= \sum_{mn} P(m - \nu, n)E(m, n, 0). \end{aligned}$$

A somewhat similar argument may be used to establish

(3.17) $$\begin{aligned} &\sum_{mn} \begin{bmatrix} f \\ f_x \\ f_y \end{bmatrix} J_{mn} E(m, n, 0) \\ &= \sum \begin{bmatrix} 1 \\ -ia(m - k) \\ -ia(n - l) \end{bmatrix} J_{kl} P(m - k, n - l)E(m, n, 0) \end{aligned}$$

where the summation for m, n, k, l on the right extends from $-\infty$ to ∞ and J_{mn} represents an arbitrary function of m and n. (3.17) is obtained by replacing m, n by k, l on the left and then introducing (3.15). The two E functions may be combined by the multiplication law for the exponential function and the right side of (3.17) obtained upon setting $m = u + k$, $n = v + l$.

Equating the coefficient of $E(m, n, 0)$ to zero in the second of equations (3.13) after using (3.16) gives

(3.18) $$B^{(1)}_{mn} = -2i\beta\gamma P(m - \nu, n).$$

The second order terms $A^{(2)}_{mn}$, $B^{(2)}_{mn}$ may be obtained by setting the values of $A^{(1)}_{mn}$, $B^{(1)}_{mn}$ in (3.14) and using (3.17):

(3.19) $$\begin{aligned} A^{(2)}_{mn} &= \sum_{kl} ia(m - k)C^{(1)}_{kl} P(m - k, n - l) \\ B^{(2)}_{mn} &= \sum_{kl} [ia(n - l)C^{(1)}_{kl} + 2\beta\gamma b(k, l)P(k - \nu, l)]P(m - k, n - l). \end{aligned}$$

Once the A's and B's are known the C's may be obtained from the divergence relation (3.6). For example

(3.20) $$C^{(1)}_{mn} = -anB^{(1)}_{mn}/b(m, n) = 2i\beta\gamma anP(m - \nu, n)/b(m, n).$$

When the appropriate expressions for the coefficients are set in (3.1) we get

(3.21) $$\begin{aligned} E_x &= -2\beta\gamma \sum_{mn} E(m, n, z) \sum_{kl} a^2(m - k)lQ(m, n, k, l) \\ E_y &= 2i \exp\{-i\beta\alpha x\} \sin \beta\gamma z - 2\beta\gamma \sum_{mn} E(m, n, z)[iP(m - \nu, n) \\ &\qquad + \sum_{kl} \{a^2(n - l)l - b^2(k, l)\}Q(m, n, k, l)] \\ E_z &= 2\beta\gamma \sum_{mn} [E(m, n, z)/b(m, n)][ianP(m - \nu, n) \\ &\qquad + \sum_{kl} \{a^3l(m^2 + n^2 - mk - nl) - anb^2(k, l)\}Q(m, n, k, l)] \end{aligned}$$

where $E(m, n, z)$ is the exponential function defined by (3.2), $(\alpha, 0, -\gamma)$ are direction cosines of the incident ray, $a = 2\pi/L$ where L is the period of the surface, ν is an integer given by $a\nu = \beta\alpha$, $\beta = 2\pi/\lambda$ and

$$Q(m, n, k, l) = P(k - \nu, l)P(m - k, n - l)/b(k, l) \tag{3.22}$$

The summations for m, n, k, l extend from $-\infty$ to $+\infty$.

The terms entering the summations in (3.21) may be divided into two classes. A term is in the first class if the corresponding values of m and n satisfy $a^2m^2 + a^2n^2 < \beta^2$ and in the second if the opposite inequality is satisfied. For a term in the first class $b(m, n)$ is positive real and $E(m, n, z)$ represents a wave traveling in the direction specified by the direction cosines am/β, an/β, $b(m, n)$ β. The corresponding electric intensity is perpendicular to the direction of propagation, as is shown by (3.6). In terms of the wavelength λ of the incident wave and the period L of the surface these direction cosines are $\lambda m/L$, $\lambda n/L$, $b(m, n)$ β. For a term in the second class $b(m, n)$ is negative imaginary and $E(m, n, z)$ corresponds to a surface wave traveling in the direction determined by m n and exponentially attenuated in the z direction.

An examination of the series for E_z shows that the terms become large for n near $\pm\beta/a = \pm L/\lambda$ and for m near zero if the coefficients around $P(-\nu, \beta/a)$ are appreciably different from zero, for then $b(m, n)$ in the denominator is small. This indicates that for some surfaces there will be an appreciable sidewise (i.e., in the y direction) scattering of the wave. It will be seen later that if the finite conductivity of the reflecting surface is taken into account the large terms remain finite even if $b(m, n) = 0$.

That E_z sometimes tends to be large may be seen from the following physical considerations. Take the case of normal incidence so that $\nu = 0$ and take the surface to be $z = 2P \cos \beta y$. The incident E_y produces a surface current in the y direction and each upward (and downward) slope of the surface may be regarded as a surface current element (infinitely long in the x direction) which radiates a field. Since the period of the surface is equal to one wavelength at the incident radiation, the E_z components of the fields of various current elements are in phase at $z = 0$ and hence the resultant E_z tends to be large.

As the roughness increases, the additional energy in the scattered radiation is obtained at the expense of the energy in the main component of the reflected wave. This is closely connected with the relation

$$\text{Real Part of } 2\beta\gamma B_{\nu 0} = \sum [\ A_{mn}^2 | + \ B_{mn}^2 \ + \ C_{mn}^2 \ ]b\ m, n \tag{3.23}$$

which is an extension of a result due to Rayleigh [4]. Here the summation extends over all values of the integers m and n such that $m^2 + n^2 < \beta^2$ a^2 (i.e., over the values for which $b(m, n)$ is real). Equation (3.23) is an exact relation and does not depend on $z = f(z, y)$ being only slightly rough. It may be established by equating to zero the average power flow through a square of side L lying on a plane z = constant parallel to the x,y-plane and at a grea height above it. $B_{\nu 0}$ is the change in the main reflected wave produced by the rough-

ness. Although (3.23) was derived by integrating Poynting's vector over square, it is interesting to note that the m,n-th term on the right is proporti to the intensity of the m,n-th component of the field times the cosine, $b(m, n$ of the angle between its direction of propagation and the z-axis. That (3 and 3.21 are in accord may be seen from the fact that (since $iP(\nu - \nu, ($ imaginary) the real part of B_ν is, from (3.21) and (3.1),

$$2\beta\gamma \sum [a^2l^2 + b\ \ k, l)] \,|\, P(k - \nu, l) \ ^2\ b(k, l) + O(f^3) \tag{3.24}$$

where the summation extend over values of k and l such that $k^2 + l^2 <$ (because b k, l is real for only these values). Furthermore,

$$|A_{mn}|^2 \quad O(f^4), \qquad B_{mn}\ ^2 \quad |\, 2\beta\gamma P(m \quad \nu, n)\,|^2 + O(f^3)$$

$$|\, C_{mn}\,|^2 = |\, 2\beta\gamma anP\ m \quad \nu, n\ \ b(m, n)\ ^2 + O(f^3)$$

and when these are put in the right hand side of (3.23) we get a result w agrees with (3.24).

Up to this point the results of this section hold for any assigned valu the $P(m, n)$'s except that they are usually required to be small. No statis considerations enter into equations (3.1) to (3.21). However, from here to end of this section we shall make use of the statistical properties of the $P(m,$ described in Section 2, to obtain various average values from the approxi expressions (3.21) for the field. From (2.3) and (3.22) follows

$$\langle Q(m, n, k, l)\rangle = \begin{cases} 0, & (m, n) \neq (\nu, 0) \\ \pi^2 W(ak - a\nu, al)\ L^2 b(k, l), & m = \nu, n = 0 \end{cases} \tag{3.25}$$

When the averages of E_x, E_y and E_z as given by (3.21) are taken only the t for which m ν, $n = 0$ remain. Furthermore, since the first power of l factor of the terms remaining in E_x and E_z and since W and $b(k, l)$ are functions of l, it may be shown that the average values of E_x and E_z va This is to be expected on physical grounds.

The average value of E_y is

$$\begin{aligned} \langle E_y\rangle &= 2i \exp\{-i\beta\alpha x\} \sin \beta\gamma z \\ &\quad + 2\beta\gamma E(\nu, 0, z) \sum_{kl} \left[\frac{a^2l^2}{b(k, l)} + b(k, l)\right] \frac{\pi^2}{L^2} W(ak - a\nu, l) \\ &\to \exp\{-i\beta(\alpha x - \gamma z)\} \\ &\quad - \exp\{-i\beta(\alpha x + \gamma z)\}\left\{1 - 2\beta \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} ds \left[\frac{\gamma s^2}{b} + \gamma b\right] \frac{W(r - \beta\alpha}{4}\right. \end{aligned} \tag{3.26}$$

where we have used $a\nu = \beta\alpha$ and have set

$$r \quad ak \quad 2\pi k/L, \qquad s = al$$

(3.27)
$$b \begin{cases} [\beta^2 \quad r \quad s^2]^{1/2}, & \beta^2 > r^2 + s^2 \\ i[r^2 + s^2 \quad \beta\,]^{1/2}, & \beta^2 < r^2 + s^2. \end{cases}$$

In going from the summation to the integration we have assumed L to approach infinity just as in (2.5). By setting

$$p = r \quad \beta\alpha, \qquad q = s$$

(3.28)
$$b = \begin{cases} [\beta \quad (p \quad \beta\alpha)^2 - q^2]^{1/2} \\ i[(p + \beta\alpha)^2 + q^2 - \beta^2]^{1\ 2} \end{cases}$$

the last term in (3 26) may be written as

(3.29)
$$2\beta \int_{-\infty}^{\infty} dp \int \quad dq \left[\frac{\gamma q^2}{b} + \gamma b\right] \frac{W(p, q)}{4}.$$

The coefficient of $\exp\{-i\beta\ \alpha x + \gamma z)\}$ in (3.26) represents the average value of the reflection coefficient and hence (3.29) represents the change in the reflection coefficient produced by the roughness.

The leading term in the mean square value of the fluctuation of E_y about the value it has in the absence of roughness is

$$\langle\ E_y \quad 2i \exp\{\quad i\beta\alpha x\} \sin\beta\gamma z \quad \rangle$$

$$4\beta\ \gamma^2 \sum_{m\ k} E^*(m, n, z)E\ k, l, z)\langle P^*(m - \nu, n)P(k - \nu, l)\rangle$$

(3.30)
$$4\beta^2\gamma \sum_{kl} \exp\{\quad z\varphi(k, l)\}\pi^2 W(ak \quad a\nu, al)\ L^2$$

$$\rightarrow 4\beta^2\gamma^2 \int_{\infty}^{\infty} dr \int_{-\infty}^{\infty} ds\, e^{-z\varphi} W(r - \beta\alpha, s)/4$$

$$= 4\beta^2\gamma^2 \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq\, e^{-z\varphi} W(p, q)/4$$

where

$$\varphi(k, l) = ib(k, l) - ib^*(k, l)$$

(3.31)
$$= \text{Imaginary part of } -2b(k, l)$$

$$= \begin{cases} 0, & k^2 + l^2 < \beta^2\ a^2 \\ 2[a^2k^2 + a^2l^2 - \beta^2]^{1/2}, & k^2 + l^2 > \beta^2\ a \end{cases}$$

and $\varphi = 0$ when $r^2 + s^2 < \beta^2$ or $(p + \beta\alpha)^2 + q^2 < \beta^2$ and

(3.32)
$$\varphi = 2[r^2 + s^2 - \beta^2]^{1/2} = 2[(p + \beta\alpha)^2 + q^2 - \beta^2]^{1/2}$$

when the inequalities are reversed. It is interesting to note that the average value of

$$E_y - 2i \exp\{-i\beta\alpha x\} \sin\beta\gamma z$$

is $O(\beta^2 f^2)$ (this is indicated by (3.26) and (3.29) since the double integral of $W(p, q)$ gives $\langle f^2(x, y)\rangle$) while the rms value of its modulus, as obtained by the square root of (3.30), is $O(\beta f)$.

When the procedure used to derive (3.30) is applied to the $O(f)$ terms in the expressions for E_z and E_x in (3.21) we obtain

(3.33)
$$\langle |E_z|^2\rangle \rightarrow 4\beta^2\gamma^2 \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \frac{e^{-z\varphi} q^2 W(p, q)}{4\,|\,b\,|^2}$$

$$\langle |E_x|^2\rangle = 0.$$

where, from (3.28),

(3.34)
$$|\,b\,|^2 = |\,\beta^2 - (p + \beta\alpha)^2 - q^2\,|.$$

Here we encounter trouble because the denominator may become zero. If $W(p, q)$ is continuous and not zero on the circle $|\,b\,|^2 = 0$ in the p, q plane, the double integral in (3.33) diverges logarithmically. This difficulty may be overcome in several ways. If the reflecting surface is not perfectly conducting a convergent double integral analogous to (3.33) may be obtained from the expression for $C_{mn}^{(1)}$ given by (7.21). When the conductivity g of the reflecting surface is large, but not infinite, equation (7.26) shows that the b occurring in the $|\,b\,|^2$ of the denominator of (3.33) should be replaced by $b + i\beta^2\ \tau$ where τ is the intrinsic propagation constant of the reflecting material: $\tau = (i\omega\mu g)^{1\ 2}$, it being assumed that the permeability μ ($\mu = 4\pi \times 10^{-7}$ henries/meter for free space) is the same for the reflecting material as for the region $z > f(x, y)$. Since b is purely real or imaginary, it is seen that the new denominator never vanishes. Another method of meeting the difficulty is to assume $f(x, y) \equiv 0$ outside a square of side L instead of taking it to be a periodic function. The integral will converge as long as L remains finite.

Equations (3.30) and (3.33) show that the first approximation to the scattered field vanishes as z approaches infinity if $W(p, q)$ is zero for the region inside the circle $(p + \beta\alpha)^2 + q^2 = \beta^2$ where φ is zero, i.e., if the average distance between the hills is rather small compared to a wavelength. This means that the reflection in this case is perfect (the modulus of the average reflection coefficient being unity).

Incidentally, as Rayleigh has pointed out, the reflection from a simple sine wave surface will be perfect if the period of the sine wave is small enough.

In order to see this from our analysis suppose the equation of the surface to be $z = 2P \cos (m_1 ax + n_1 ay)$ so that all of the $P(m, n)$'s are zero except

$$P(m_1, n_1) = P(-m_1, -n_1) = P = \text{real}.$$

The only non-vanishing $O(f)$ terms in (3.21) are the two given by $m = \nu \pm m_1$, $n = \pm n_1$ (e.g., the upper signs go together), and the only non-vanishing $O(f^2)$ terms are the three given by $m = \nu \pm 2m_1$, $n = \pm 2n_1$ and $m = \nu$, $n = 0$. It follows that if m_1 and n_1 are such that

$$(|\nu| - |m_1|)^2 + n_1^2 > \beta^2/a^2 \tag{3.35}$$

the only term which can possibly correspond to a scattered wave is the one given by $m = \nu$, $n = 0$ (remember that $|\nu| < \beta/a$) because all of the others correspond to surface waves which carry no energy away from the surface. Since the $m = \nu$, $n = 0$ term corresponds to a wave traveling in the same direction as the main reflected wave it cannot be regarded as scattering. All it can do is change the phase of the reflection coefficient. Our work doesn't go beyond $O(f^2)$ terms but it doesn't seem likely that the higher order approximations will bring in any terms which can be interpreted as scattering.

However, the situation is quite different if the surface consists of the sum of two (or more) rapidly varying sine waves whose "interference pattern" has a period long enough to produce scattering. For example, let the surface be

$$z = 2P_1 \cos (m_1 ax + n_1 ay) + 2P_2 \cos (m_2 ax + n_2 ay) \tag{3.36}$$

where m_1, n_1 satisfy (3.35) and m_2, n_2 satisfy a similar inequality. An examination of (3.21) and the definition of $Q(m, n, k, l)$ shows that the $O(f^2)$ terms which might produce scattering are the two for which $m = \nu \pm (m_1 - m_2)$, $n = \pm(n_1 - n_2)$. At least one of these is certain to produce scattering if

$$(|\nu| - |m_1 - m_2|)^2 + (n_1 - n_2)^2 < \beta^2/a^2. \tag{3.37}$$

because it would correspond to a wave for which $b(m, n)$ is real and hence would carry energy away from the surface in a direction different from that of the main reflected wave. Even if (3.37) were not satisfied there is a possibility of higher order terms corresponding to scattering.

If we now consider the case of the rough surface with the above examples in mind we see that although the reflection may sometimes be perfect to a first approximation, the $O(f^2)$ terms in (3.21) give rise to a scattered field (somewhat similar to the Rayleigh scattering produced by small particles) which does not vanish as z becomes large. In order to study mean square values involving the $O(f^2)$ terms it is necessary to deal with averages of expressions containing the product of four $P(m, n)$'s. Since the results appear to be rather complicated, we shall not go farther than to state the following result which may be applied to our problem when P_n is replaced by $P(m, n)$ and the summation taken with respect to m, n instead of n, and likewise for k, n', k'.

Let P_0 be real. Let P_0 and the real and imaginary parts of P_1, P_2, $\cdots$

be independent random variables with average value zero. Let the real and imaginary parts of P_n, $n > 0$, have the same mean square value so that $\langle P_n^2 = 0$ unless $n = 0$, and define $P_{\ n}$ as the conjugate complex of P_n so that

$$\langle P_{\ n} P_n \rangle \qquad P_n{}^2 \qquad \langle\ P_n^2 |\rangle,$$

$$\langle P_m P_n \rangle \quad 0 \qquad \text{if} \qquad m \neq -n.$$

If $F(n, k, n', k')$ denotes an arbitrary function of n, k, n', k' it can be shown that, if the summations run from $\ \infty$ to $+\infty$,

$$\begin{aligned}
&\sum_{nkn'k'} F(n, k, n', k') \langle P_n P_k P_{n'} P_{k'} \\
&\quad = \sum_{nk} [F(n, -n, k, -k) + F(n, k, \ \ n, -k) + F(n, k, -k, -n)] \\
&\qquad \cdot \langle | P_n{}^2 \rangle \langle | P_k^2 | \rangle \\
&\qquad + \sum_n [F(n, -n, n, \ \ n) + F(n, n, -n, -n) + F(n, -n, -n, n)] \\
&\qquad \cdot [\langle | P_n^4 | \rangle - 2(\langle | P_n^2 \ \rangle)^2] \\
&\qquad + F(0, 0, 0, 0)[3(\langle P_0^2 \rangle)^2 - 2\langle P_0^4 \rangle].
\end{aligned} \tag{3.38}$$

One method of establishing this result is to break the four-fold summation into the subgroups for which (1) $k \neq n$, $k \neq -n$, (2) $k = n$, $n \neq 0$, (3) $k = -n$, $n \neq 0$, (4) $k = 0$, $n = 0$. The terms which have averages different from zero in subgroup (1) are those for which (1a) $n' = -n$, $k' = -k$, (1b) $n' = -k$, $k' = -n$. Likewise for the other groups we have (2a) $n' = k' = -n$, (3a) $n' = -n$, $k' = n$, (3b) $n' = n$, $k' = -n$, (3c) $n' = -k'$ but $n' \neq \pm n$, (4a) $n' = k' = 0$, (4b) $n' = -k'$, $n' \neq 0$.

When, as in the case of the rough surface, the surface $z = f(x, y)$ has many Fourier components of the same order of magnitude, the only term of importance on the right hand side of (3.38) is the double summation over n and k. This term goes into a fourfold integral involving the product of two $W(p, q)$ functions.

4. *Incident Wave Vertically Polarized*

In this section we assume the electric intensity of the field to be, in the absence of roughness,

$$\begin{aligned}
E_x^a &= 2i\gamma \exp\{-i\beta\alpha x\} \sin \beta\gamma z, \qquad E_y^a = 0 \\
E_z^a &= 2\alpha \exp\{-i\beta\alpha x\} \cos \beta\gamma z
\end{aligned} \tag{4.1}$$

where the symbols have the same meaning as in Section 3. In particular,

$\beta = 2\pi/\lambda$, and $(\alpha, 0, -\gamma)$, $(\alpha, 0, \gamma)$ are the direction cosines of the incident and reflected rays, respectively. A procedure similar to that used to obtain (3.21) leads to the following expressions, accurate to $O(f^2)$ terms, for the el ctric intensity in the presence of the slightly rough surface.

$$E_x = E_x^a + 2 \sum_{mn} E(m, n, z)[i(\alpha am \quad \beta P(m \quad \nu \; n)$$

$$+ \sum_{kl} \{a^2(m-k)(\nu \quad k\,\beta + (\beta \quad \alpha am\, b^2\, k, l\,\}Q(m, n, k, l)]$$

$$E_y = 2a \sum_{mn} E(m, n, z)[i\alpha n P(m-\nu, n$$

$$+ \sum_{kl} \{a(n-l)(\nu-k)\beta - \alpha n b^2(k, l)\}Q(m, n, k, l)]$$

(4.2)
$$E_z = E_z^a + 2 \sum_{mn} [E(m, n, z) \; b(m, n)]$$

$$\cdot \Big[i\{a(m-\nu)\beta + \alpha b^2(m, n)\}P(m-\nu, n)$$

$$+ \sum_{kl} \{a^3(k-\nu)(m^2+n^2 \quad mk - nl)\beta$$

$$+ a[\alpha a(m^2+n^2) - m\beta]b^2(k, l)\}Q(m, n, k, l)\Big].$$

In these equations $E(m, n, z)$ is the exponential function of x, y, z defined by (3.2) and $Q(m, n, k, l)$ is the function (3.22) containing the product of two P's.

The average electric intensity of the reflected wave is in the direction specified by the direction cosines $(-\gamma, 0, \alpha)$. The corresponding wave function approaches, as $L \to \infty$, $E(\nu, 0, z)$ multiplied by

(4.3)
$$1 - 2\beta \int_{\infty}^{\infty} dr \int^{\infty} ds \left[\frac{(r-\beta\alpha)^2}{\gamma b} + \gamma b\right] W(r-\beta\alpha, s) \; 4$$

$$= 1 - 2\beta \int \; dp \int \; dq \left[\frac{p^2}{\gamma b} + \gamma b\right] W(p, q \quad 4$$

where b is defined as a function of r, s and p, q by equations (3.27) and (3.28). The derivation of (4.3) is similar to that of its analogue in (3.26): the expressions (4.2) are averaged, the reflected wave picked out, and the square root of the sum of the squares of its x, y, z components taken (the average y component turns out to be zero).

An idea of how the field components vary about their values in the absence of roughness may be obtained from the following analogues of (3.30) and (3.33).

$$\langle | E_x - E_x^a |^2 \rangle = 4 \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq\, e^{-z\varphi}(\alpha p - \beta\gamma^2)^2 W(p, q)/4$$

(4.4)
$$\langle | E_y |^2 \rangle = 4 \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq\, e^{-z\varphi} \alpha^2 q^2 W(p, q)/4$$

$$\langle | E_z - E_z^a |^2 \rangle = 4 \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq\, e^{-z\varphi}(p\beta + \alpha b^2)^2 W(p, q)/4 \mid b \quad .$$

Here, as in (3.33), the last integral may not converge on the circle $b = 0$. It was pointed out that this difficulty can be overcome in the case of horizontal polarization by considering the electrical properties of the reflecting surface, and the same is probably also true for the case of vertical polarization.

The analogue of (3.23) is

(4.5)
$$\text{Real Part of } 2\beta A_{\nu 0} = \sum [| A_{mn}^2 | + | B_{mn}^2 | + | C_{mn}^2 |] b(m, n)$$

where the summation extends only over those values of m and n for which $b(m, n)$ is real $(m^2 + n^2 \le \beta^2/a^2)$ and A_{mn} etc. are defined by

$$E_x = E_x^a + \sum A_{mn} E(m, n, z)$$

$$E_y = \sum B_{mn} E(m, n, z)$$

$$E_z = E_z^a + \sum C_{mn} E(m, n, z).$$

Equation (4.5) and $C_{\nu 0} = -\alpha A_{\nu 0}/\gamma$, which follows from the divergence relation (3.6), give a partial check on equations (4.2).

5. *Special Cases*

Suppose that the roughness spectrum, $W(p, q)$ is zero except for a small region around $p = 0$, $q = 0$. In this case the average distance between the hills of the surface is large compared to the wavelength of the incident radiation. The function b defined by (3.28) differs but little from its value at $p = q = 0$, namely $\beta\gamma$, and (3.29) becomes

(5.1)
$$2\beta \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq\, \gamma\beta\gamma W(p, q) \;\; 4 = 2\beta^2\gamma^2 \langle f^2(x, y) \rangle.$$

Here we have used expression (2.5) for the mean square value of $f(x, y)$ and have assumed $\gamma q^2/b$ in (3.29) to be negligibly small in the region where $W(p, q)$ is different from zero. The average value of the reflection coefficient for horizontal polarization now becomes

(5.2)
$$1 - 2\beta^2\gamma^2 \langle f^2(x, y) \rangle.$$

A similar treatment of (4.3) which involves the neglect of $p^2/\gamma b$ shows that 5.2) also holds for the case of vertical polarization. It is interesting to note that (5.2) agrees with the first two terms in the expansion of a result obtain y W. S. Ament in which βf is not required to be small namely, that the roughness reduces the amplitude of the average reflected waves by the fa tor $\exp\{-2\beta^2\gamma^2\langle f^2(x, y)\rangle\}$. As pointed out in the introduction, this agreement is all that the approximate nature of our results will allow.

When $W(p, q)$ differs from zero only in the region around $p \quad 0, q = 0$, equations (3.30) and (3.33) show that for horizontal polarization

$$\langle| E_y - 2i \exp\{-i\beta\alpha x\} \sin \beta\gamma z |^2\rangle \quad 4\beta^2\gamma^2 f^2(x, y)\rangle, \tag{5.3}$$

$$\langle E_z |^2\rangle = 4 f_y^2(x, y\rangle$$

where $f_y(x, y) = \partial f(x, y)/\partial y$; equations (4.4) show that for vertical polarization

$$\langle| E_x - E_x^s |^2\rangle = 4\beta^2\gamma^4 f^2(x, y)$$

$$\langle| E_y |^2\rangle = 4\alpha^2\langle f_y^2(x, y)\rangle \tag{5.4}$$

$$\langle| E_z - E_z^s | \quad = 4\beta^2\gamma^2\alpha^2\langle f^2(x, y)\rangle.$$

Suppose now that $W(p, q)$ is such as to make the terms $\gamma q^2/b$ and $p^2 \gamma b$, which were neglected above, the dominant terms in the integrands of (3.29) and (4.3). The average distance between hills of the corresponding surface will n y be small compared to a wavelength. The magnitudes of the average reflection coefficients are then approximately

$$1 - 2\beta \int_{-\infty}^{\infty} dp \int dq\, \gamma q^2 W(p, q)/4b \quad 1 \quad \gamma s_p$$

$$1 - 2\beta \int dp \int dq\, p W(p, q \quad 4\gamma b \quad 1 - \frac{s_h}{\gamma} \tag{5.5}$$

for horizontal and vertical polarizations, respectively. Here s_p and s_h stand for small quantities, and γ is the cosine of the angle between the z-axis and the reflected ray. The remarkable thing about the reflection coefficients (5.5) is that they depend on γ in the same way as do the corresponding reflection coefficients, computed from Fresnel's formulas, for a good, but not perfect, plane conductor.

For vertical incidence $\gamma = 1$, $\alpha \quad 0$ and the two expressions given by (5.5) reduce to essentially the same thing, the q^2 in the first expression (where the incident E is parallel to the y-axis) goes over to the p^2 in the second expression (where the incident E is parallel to the x-axis) because of the difference in the assumed incident waves.

6. *Propagation along Surface*

As the condition of grazing incidence is approached γ appr aches zero, and expression (4.3) giving the average reflection coefficient for vertical po rization breaks down. n his case a modification of th method used to study reflection may be used to obtain a solution corresponding to a wave guided by the surface. A solution of this sort is to be expected since it has been known for some time that a corrugated or slotted surfa e will support a typical "surface wave" in which the field decreases exponentially with distance from the surface.

To start with, we take the perfectly conducting surface to be

$$z \quad 2P \cos sx = f \tag{6.1}$$

which shows that f is now merely a function of x. Guided by the known properties of surface waves, we assume that there exists a wave in which the electric intensity is predominantly in the z direction (approximately normal to the surface) and that there is also a small component of E in the x direction (in the direction of propagation). We also tacitly assume that the velocity of propagation of the wave does not differ much from that of a wave traveling freely in the medium above the surface; i e., if the propagation of the principal part of the wave is described by $\exp\{i\omega t - ihsx\}$ then hs approaches $\beta = 2\pi \ \lambda$ as the amplitude P of the corrugations approaches zero.

When we attempt to express our assumptions as equations some experimentation suggests the forms

$$E_x = \sum_m A_m E(h + m, z), \qquad E_y = 0$$

$$E_z = E(h, z) + \sum_m C_m E(h + m, z) \tag{6.2}$$

where the summations with respect to m extend over all integers from $-\infty$ to ∞ and A_m, C_m are small quantities which approach zero with P. In order to fix the amplitud of e various components C_0 is taken to be zero so that t ere is no term correspondin to $E(h, z)$ in summation rt f E_z. Here

$$E(h + m, z) = \exp\{-i(h + m)sx - ib(h + m)z\}$$

$$b(h + m) = \begin{cases} [\beta^2 - (h + m)^2 s^2]^{1/2}, & \beta^2 > (h + m)^2 s^2 \\ -i[(h + m)^2 s^2 - \beta^2]^{1/2}, & \beta^2 < (h + m)^2 s^2 \end{cases} \tag{6.3}$$

so that the components (6.2) satisfy the wave equation. Since the difference between β and hs is assumed to be small it follows that $b(h)$ is to be regarded as small.

Since we do not intend to carry our approximations beyond $O(\beta^2 f^2)$ we may use the first of the boundary conditions (3.10) which, for our surface (6.1), becomes

$$E_x = N_x E_z = -f_x E_z = 2Ps \sin sx E_z \tag{6.4}$$
$$= Ps(-ie^{isx} + ie^{-isx})E_z \, .$$

This relation must be satisfied at $z = 2P \cos sx = f$ to within an accuracy of $O(P^2)$.

Upon substituting the assumed expressions (6.2) for E_x and E_z in the boundary condition (6.4), using relations of the form

$$A = A_m^{(1)} + A_m^{(2)} + \cdots$$

$$E(h + m, f) = E(h + m, 0)[1 - ib(h + m)f + \cdots]$$

$$E(h + m, 0)f = PE(h + m - 1, 0) + PE(h + m + 1, 0)$$

in the same manner as in the reflection problem, and equating first order terms we see that

$$A_1^{(1)} = Psi, \qquad A_{-1}^{(1)} = -Psi \tag{6.5}$$
$$A_m^{(1)} = 0 \qquad \text{if} \qquad m \neq 1 \text{ or } -1.$$

The divergence relation div $E = 0$ gives

$$(h + m)sA_m + b(h + m)C_m = 0, \qquad m \neq 0 \tag{6.6}$$
$$hsA_0 + b(h) = 0, \qquad m = 0.$$

Since $A_0^{(1)}$ is zero, $b(h)$ is smaller than a first order term (it will be shown later to be $O(P^2)$). From the first of equations (6.6) it follows that

$$C_1^{(1)} = -(h + 1)sA_1^{(1)}/b(h + 1), \qquad C_{-1}^{(1)} = -(h - 1)sA_{-1}^{(1)} \, b(h - 1) \tag{6.7}$$
$$C_m^{(1)} = 0 \qquad \text{if} \qquad m \neq 1, 0 \text{ or } -1.$$

Equating the second order terms in (6.4), and using (6.5) and (6.7) gives

$$A_0^{(2)} = iP[b(h + 1)A_1^{(1)} - sC_1^{(1)} + b(h - 1)A_{-1}^{(1)} + sC_{-1}^{(1)}]$$
$$= P^2 s\left[\frac{hs^2 - \beta^2 + h^2s^2}{b(h + 1)} + \frac{hs^2 + \beta^2 - h^2s^2}{b(h - 1)}\right]$$
$$A_2^{(2)} = iP[b(h + 1)A_1^{(1)} + sC_1^{(1)}] \tag{6.8}$$
$$A_{-2}^{(2)} = iP[b(h - 1)A_{-1}^{(1)} - sC_{-1}^{(1)}]$$
$$A_m^{(2)} = 0 \qquad \text{if} \qquad m \neq 0, 2 \text{ or } -2.$$

The expression for $A_0^{(2)}$ is of particular importance because when it is combined

with the second of equations (6.6), which we write as $A_0^{(2)} = -b(h) \, hs$, we obtain an equation which may be solved for the propagation constant hs in the x direction:

$$-b(h) = P^2hs^2\left[\frac{hs^2 - \beta^2 + h^2s^2}{b(h + 1)} + \frac{hs^2 + \beta^2 - h^2s^2}{b(h - 1)}\right]. \tag{6.9}$$

This expression shows that $b(h)$ is $O(P^2)$ and therefore, when P is small in accordance with our assumptions, hs is nearly equal to β. Replacing hs by β in (6.9) gives

$$b(h) \approx -iP^2\beta^2s[(1 + 2\beta/s)^{-1/2} + (1 - 2\beta/s)^{-1/2}] \tag{6.10}$$

which shows that if $s > 2\beta$, $b(h)$ is negative imaginary and $E(h, z)$ decreases exponentially with increasing z. Thus in this case we have a true surface wave.

When s is much greater than β so that the surface has many corrugations in one wavelength of the electromagnetic wave, we get from (6.10)

$$b(h) \approx -2iP^2\beta^2s$$

$$hs \approx \beta + 2P^4\beta^3s^2$$

and the principal part of the field is the surface wave

$$E_z \approx \exp\{-i\beta(1 + 2P^4\beta^2s^2)x - 2P^2\beta^2sz\} \tag{6.11}$$

which travels a little more slowly than a free wave.

The same type of analysis may be to investigate the surface wave which is guided by the mor general rough surface described in Section 2. We assume

$$E_x = \sum_{mn} A_{mn}E(m + h, n, z) \tag{6.12}$$
$$E_y = \sum_{mn} B_{mn}E(m + h, n, z)$$
$$E_z = E(h, 0, z) + \sum_{mn} C_{mn}E(m + h, n, z)$$

where the summations extend over all integral values of m and n between plus and minus infinity, $C_{00} = 0$, and $E(m + h, n, z)$ is defined by (3.2) and (3.3) with m replaced by $m + h$. The situation is somewhat similar to putting $\gamma = 0$, $\alpha = 1$ in the vertical polarization case of reflection. The boundary conditions are

$$E_x = -f_xE_z \, , \qquad E_y = -f_yE_z \tag{6.13}$$

and these, together with the condition div $E = 0$:

$$a(m+h)A_{mn} + anB_{mn} + b(h+m, n)C_{mn} = 0, \qquad m, n \neq 0$$
(6.14)
$$ahA_{00} + b(h, 0) = 0, \qquad m = n = 0$$

lead to the expressions

$$A^{(1)}_{mn} = iamP(m, n), \qquad B^{(1)}_{mn} = ianP(m, n)$$
(6.15)
$$C^{(1)}_{mn} = -ia^2[m^2 + mh + n^2]P(m, n)/b(h + m, n)$$

for the $O(f)$ terms in the coefficients. The $O(f^2)$ terms in A_{mn} and B_{mn} are

$$A^{(2)}_{mn} = \sum_{kl} i[a(m-k)C^{(1)}_{kl} + b(h+k, l)A^{(1)}_{kl}]P(m-k, n-l)$$
(6.16)
$$B^{(2)}_{mn} = \sum_{kl} i[a(n-l)C^{(1)}_{kl} + b(h+k, l)B^{(1)}_{kl}]P(m-k, n-l).$$

Since $A^{(1)}_{0}$ is zero, from (6.15), $A^{(2)}$ is given by the second of equations (6.14). Equating this to the value of $A^{(2)}_{00}$ given by (6.16) leads to

(6.17) $$b(h, 0) = \sum_{kl} a^2hk(\beta^2 - a^2h^2 - a^2hk)\, |P(k, l)|^2 / b(h+k, l).$$

As the roughness decreases, $b(h, 0)$ approaches zero and ha approaches β and we have

$$b(h, 0) \approx -\sum_{l} a^2\beta^2k^2\, |P(k, l)|^2 / b(k + \beta/a, l).$$

When this is averaged over the universe of rough surfaces mentioned in Section 2 and when (2.3) is used we obtain, upon letting L approach infinity,

(6.18) $$b(h, 0) \approx -\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq\, \beta^2p^2W(p, q)/4b_1$$

where b_1 is the function of p and q obtained by setting $\alpha = 1$ in expression (3.28) for b:

(6.19) $$b_1 = \begin{cases} [\beta^2 - (p+\beta)^2 - q^2]^{1/2} \\ -i[(p+\beta)^2 + q^2 - \beta^2]^{1/2}. \end{cases}$$

The principal part of the surface wave is

$$E_s = \exp\{-iahx - ib(h, 0)z\}$$

which leads us to introduce $B = ib(h, 0)$ so that

$$B^2 = -\beta^2 + a^2h^2 \approx (ah - \beta)2\beta$$

$$ah \approx \beta + B^2/2\beta.$$

We may therefore summarize our result by saying that the principal part of the surface wave corresponding to the general (slightly rough) surface of Section 2 is

$$E_s = \exp\{-i\beta(1 + B^2/2\beta^2)x - Bz\}$$

where

$$B = B_r + iB_i = -i\beta^2 \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq\, p^2W(p, q)/4b_1$$

and the attenuation in the x direction is $-B_rB_i/\beta$ (nepers/meter). The definition (6.19) of b_1 shows that B_i is never positive. It also shows that if $W(p, q)$ is zero where b_1 is real, namely inside the circle of radius β centered at $p = -\beta$, $q = 0$, B_i is zero and there is no attenuation. This corresponds to the case where the hills of the surface are close together and is ın agreemen wıt t e vıew that the guiding action of the surface is due to rapı y undulating components o $z = f(x, y)$ while the attenuatıon ıs ue e scattering produced b the more slowly varying components. It should e remembered that (6.18) is on y an approximate expression for $b(h, 0)$. It seems probable that more accurate expressions would show an attenuation even if $W(p, q)$ were zero in the circle mentioned above because this is no guarantee that $A^{(2)}_{mn}$ and $B^{(2)}_{mn}$ given by (6.16) will vanish for values of m and n which correspond to waves carrying energy away from the surface. Thus it appears that even though the surfaces $z = P \cos sx$ and $z = Q \cos tx$ can carry surface waves without attentuation when $s > 2\beta$ and $t > 2\beta$, the same is not true of the surface $z = P \cos sx + Q \cos tx$ if, for example, $s - t$ were almost equal to β. The situation is somewhat similar to the one encountered in the discussion of reflection from the surface (3.36).

7. *Reflection from Wavy Interface between Two Media—Horizontal Polarization*

Let the interface coincide approximately with the plane $z = 0$ and let the propagation constants σ and τ of the upper ($z > 0$) and lower media, respectively, be given by

(7.1) $$\sigma = i\omega(\mu\epsilon_0)^{1/2} = i\beta, \qquad \tau = \sigma(\epsilon_r + g/i\omega\epsilon_0)^{1/2}.$$

Here we have assumed tha both m have the same ermea that the ratio of their dielectric constants is ϵ_r. g is e conductivity of the lower medium and ϵ_0 the dielectric constant of the upper medium. The upper medium is non-conducting. For free space μ 1.257 × 10^{-6} henry/meter and ϵ_0 = 8.854 × 10^{-12} farad/meter.

If the interface coincided exactly with the plane $z = 0$ the electric intensity for horizontal polarization would be

$$
\begin{aligned}
&E_y = E^+ \equiv \exp\{-\sigma\alpha x\}(\exp\{\sigma\gamma z\} + R\exp\{-\sigma\gamma z\}), && z > 0\\
&E_y = E^- \equiv T\exp\{-\sigma\alpha x + \tau\gamma' z\}, && z < 0
\end{aligned}
\tag{7.2}
$$

$$
\tau\alpha' = \sigma\alpha = i a\nu, \qquad \gamma' = (1 - \alpha'^2)^{1/2}
$$

$$
R = \frac{1 - \dfrac{\tau\gamma'}{\sigma\gamma}}{1 + \dfrac{\tau\gamma'}{\sigma\gamma}} \qquad T = \frac{2}{1 + \dfrac{\tau\gamma'}{\sigma\gamma}}.
$$

As before, $\alpha = \sin\theta$ and $\gamma = \cos\theta$ where θ is the angle between the z-axis and the reflected ray.

When the equation of the separating surface is $z = f(x, y) \equiv f$ we assume the electric intensity to be

$$
\begin{aligned}
E_x &= \begin{cases} \sum A_{mn}E(m, n, z) & \text{for} \quad z > f \\ \sum G_{mn}F(m, n, z) & \text{for} \quad z < f \end{cases}\\
E_y &= \begin{cases} E^+ + \sum B_{mn}E(m, n, z) & \text{for} \quad z > f \\ E^- + \sum H_{mn}F(m, n, z) & \text{for} \quad z < f \end{cases}\\
E_z &= \begin{cases} \sum C_{mn}E(m, n, z) & \text{for} \quad z > f \\ \sum I_{mn}F(m, n, z) & \text{for} \quad z < f \end{cases}
\end{aligned}
\tag{7.3}
$$

where E^+, E^- are given by (7.2) (with the dividing surface $z = 0$ replaced by $z = f(x, y)$) and

$$
\begin{aligned}
E(m, n, z) &= \exp\{-ia(mx + ny) - ib(m, n)z\}\\
F(m, n, z) &= \exp\{-ia(mx + ny) + ic(m, n)z\}\\
ib(m, n) &= [\sigma^2 + a^2(m^2 + n^2)]^{1/2} \qquad a = 2\pi/L\\
ic(m, n) &= [\tau^2 + a^2(m^2 + n^2)]^{1/2}.
\end{aligned}
\tag{7.4}
$$

Here $b(m, n)$ is the same as the $b(m, n)$ defined by (3.3) and is either positive real or negative imaginary. The same would be true of $c(m, n)$ if the lower medium were non-conducting.

At $z = f$ we require the continuity of

$$
\begin{aligned}
&E_x - N_x(N_xE_x + N_yE_y + N_zE_z)\\
&E_y - N_y(N_xE_x + N_yE_y + N_zE_z)
\end{aligned}
\tag{7.5}
$$

and two other expressions obtained by substituting H (the magnetic intensity) for E. When we assume the components N_x, N_y of the normal to be small (so that $N_z \approx 1$), and also assume E_x and E_z to be small, (7.5) becomes

$$
\begin{aligned}
&E_x - N_xN_yE_y - N_xE_z\\
&(1 - N_y^2)E_y - N_yE_z\,.
\end{aligned}
\tag{7.6}
$$

The H conditions corresponding to (7.5) and the assumptions that N_x, N_y, E_x, E_y and their derivatives must be small tell us that the two expressions

$$
\begin{aligned}
&\frac{\partial E_z}{\partial y} - (1 - N_x^2)\frac{\partial E_y}{\partial z} - N_x\frac{\partial E_y}{\partial x} + N_x\frac{\partial E_x}{\partial y}\\
&\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} + N_yN_x\frac{\partial E_y}{\partial z} - N_y\frac{\partial E_y}{\partial x} + N_y\frac{\partial E_x}{\partial y}
\end{aligned}
\tag{7.7}
$$

must be continuous at $z = f$. Here we have made use of the assumption that the two media have the same permeability, and have neglected $O(f^2)$ terms. The terms $N_x^2\,\partial E_y/\partial z$ and $N_yN_x\,\partial E_y/\partial z$ may be omitted from (7.7) since the first of the two relations

$$
\begin{aligned}
\sigma\gamma(1 - R) &= T\tau\gamma'\\
1 + R &= T
\end{aligned}
\tag{7.8}
$$

ensures the continuity (out to $O(f^2)$) of the terms in question. In the same way, the second of relations (7.8) enables us to omit $N_xN_yE_y$ and $N_y^2E_y$ from (7.6).

When the assumed expressions (7.3) for the electric intensity are set in the boundary conditions (7.6) and (7.7), as just amended, the terms arising from E^+ and E^- can be simplified by using (7.8). For example, in the second of equations (7.6) these terms are

$$
\begin{aligned}
&\exp\{-\sigma\alpha x\}(\exp\{\sigma\gamma f\} + R\exp\{-\sigma\gamma f\} - T\exp\{\tau\gamma' f\})\\
&\qquad = \exp\{-\sigma\alpha x\}f^2U + O(f^3)
\end{aligned}
\tag{7.9}
$$

where

$$
U = T(\sigma^2 - \tau^2)/2. \tag{7.10}
$$

After similar reductions are made in (7.7), the four relations arising from (7.6) and (7.7) may be written as

$$
\begin{aligned}
&\sum\{[A_{mn} + f_xC_{mn}]E(m, n, f) - [G_{mn} + f_xI_{mn}]F(m, n, f)\} = 0\\
&\exp\{-\sigma\alpha x\}f^2U + \sum\{[B_{mn} + f_yC_{mn}]E(m, n, f)\\
&\qquad - [H_{mn} + f_yI_{mn}]F(m, n, f)\} = 0\\
&-\exp\{-\sigma\alpha x\}U[2f + \tau\gamma' f^2] + i\sum\{[-anC_{mn} + b(m, n)B_{mn}
\end{aligned}
\tag{7.11}
$$

$$- f_z amB_m + f a\imath\ 1_m]E\ n\ , n,\ f)$$

$$- [-anI_m \quad c\ n\ , n\ H_m \quad f_z amH\ _m + f_z anG_m]F(m, n, f\ \} \quad 0$$

$$\sum \{[-b(m, n\ A_{mn} + amC \quad f\ amB_m + f_y an\ 4_m]E\ m\ n\ f$$

$$- [c(m, n\ G_{mn} + amI_m \quad f_y amH_{mn} + f_y a\imath\ G_m]F\ m, n, f\ \} \quad 0$$

where $O(f^3)$ terms are neglected.

We now assume α is such that $\sigma\alpha = ia\nu$ where ν is an integer. In order to separate the first and second order terms in (7.11) we write the various coefficients as $A^{(1)}_{mn} + A^{(2)}_{mn} + \cdots$, and so on and use the approximate expressions

(7.12)
$$E(m, n, f) \quad [1 \quad ib\ m\ n)f]E\ m, n, 0$$

$$F(m, n, f) \quad [1 + ic\ m, n)f]E\ m, n, 0\ .$$

By replacing f exp $\{-ia\nu x\}$ by it Fourier series expansion (3.16) and proceeding as in Section 3 we find that the first order terms in 7.11) lead to

$$A^{\ 1}_{\ m} \quad G^{\ 1}_{\ n}, \quad B^{(1)}_{mn} = H^{(1)}_{mn}$$

(7.13)
$$id(m, n)B^{\ 1}_{\ mn} - ian\ C^{\ 1}_{\ mn} \quad I^{(1)}_{m} = 2UP(m \quad \nu, n$$

$$-d\ m, n)A^{\ 1}_{\ m} + am\ C^{(1)}_{mn} - I^{\ 1}_{\ m}\ ) = 0$$

where

(7.14)
$$d\ m \quad = b\ m, n) + c(m, n).$$

The equations arising from the second order terms in the first two of equations (7.11) may be simplified with the help of equation (3.17), the relations (7.13) between the first order terms, and the expansion

(7.15)
$$\exp\ \{-ia\nu x\}f^2 = \sum P(k - \nu, l)P(m \quad k, n - l)E\ m, n, 0$$

where the summation on the right extends over all integral values of m, n, k, l from $-\infty$ to $+\infty$. In dealing with the last two equations of 7.11) we need the additional results

(7.16)
$$c^2(m, n) - b^2(m, n) \quad \sigma^2 \quad \tau$$

$$b(m, n)C^{(1)}_{mn} + c(m, n\ I^{\ 1} \quad 0$$

the first of which follows from the definitions of $c\ m, \tau)$ and $b\ m, n$ and the second from subtraction of the first order terms in th two div E 0 equation

(7.17)
$$amA_{mn} + anB_{mn} + b(m, n\ C_m \quad 0$$

$$amG_{mn} + anH_{mn} - c(m\ n\ I_m \quad 0.$$

The results of this simplification are given by the equations

(7.18)
$$A^{(2)}_{mn} - G^{(2)}_{mn} = h_1$$

$$B^{(2)}_{mn} - H^{(2)}_{mn} = h_2$$

$$an(C^{(2)}_{mn} - I^{(2)}_{mn}) - b(m, n)B^{(2)}_{mn} - c(m, n)H^{(2)}_{mn} = h_3$$

$$am(C^{(2)}_{mn} - I^{(2)}_{mn}) - b(m, n)A^{(2)}_{mn} - c(m, n)G^{(2)}_{mn} = h_4$$

where, taking the summations over k and l,

(7.19)
$$h_1 = iam \sum (C^{(1)}_{kl} - I^{(1)}_{kl})P(m - k, n - l)$$

$$h_2 = \sum [UP(k - \nu, l) + ian(C^{(1)}_{kl} - I^{(1)}_{kl})]P(m - k, n - l)$$

$$h_3 = i \sum [U\tau\gamma' P(k - \nu, l) + (\sigma^2 - \tau^2)B^{(1)}_{kl}]P(m - k, n - l)$$

$$h_4 = i(\sigma^2 - \tau^2) \sum A^{(1)}_{kl}P(m - k, n - l).$$

Equations (7.13), (7.17) and (7.18) may now be used to obtain expressions, valid as far as $O(f^2)$, for the coefficients. From (7.16)

(7.20)
$$I^{(1)}_{mn} = -\frac{b(m, n)}{c(m, n)} C^{(1)}_{mn}, \qquad C^{(1)}_{mn} - I^{(1)}_{mn} = \frac{d(m, n)}{c(m, n)} C^{(1)}_{mn}$$

and these relations enable us to derive the expressions

(7.21)
$$A^{(1)}_{mn} = G^{(1)}_{mn} = \frac{i2Ua^2mnP(m - \nu, n)}{d(m, n)D_{mn}}$$

$$B^{(1)}_{mn} = H^{(1)}_{mn} = \frac{i2UP(m - \nu, n)}{d(m, n)} \left[\frac{a^2n^2}{D_{mn}} - 1\right]$$

$$C^{(1)}_{mn} = \frac{i2Uan\ c(m, n)P(m - \nu, n)}{d(m, n)D_{mn}}$$

$$C^{(1)}_{mn} - I^{(1)}_{mn} = \frac{i2UanP(m - \nu, n)}{D_{mn}}$$

where

(7.22)
$$D_{mn} = a^2(m^2 + n^2) + b(m, n)c(m, n).$$

Explicit expressions for the h's are obtained when (7.21) is put in (7.19). The second order terms may be obtained from

(7.23)
$$dDA^{(2)}_{mn} = a^2m^2bh_1 + (D - a^2m^2)(ch_1 - h_4) + a^2mn(bh_2 - ch_2 + h_3)$$

$$dDB^{(2)}_{mn} = a^2n^2bh_2 + (D - a^2n^2)(ch_2 - h_3) + a^2mn(bh_1 - ch_1 + h_4)$$

$$dDC^{(2)}_{mn} = \tau^2a(mh_1 + nh_2) + ca(mh_4 + nh_3)$$

by dividing through by dD (where we have written d, b, c, and D for $d(m, n)$, $b(m, n)$, $c(m, n)$, and D_{mn}). These expressions are obtained from (7.18) and (7.17) (written out for the second order terms).

The manner in which these expressions approach the earlier expressions for the perfect conductor may be examined by letting the conductivity g approach infinity. From (7.1) we see that, since $\sigma \quad i\beta$, τ behaves like a large positive number multiplied by $i^{1/2}$. From equations (7.2, 4, 10, 14, and 22)

$$\alpha' = \sigma\alpha/\tau, \qquad \gamma' \quad 1 + O(\tau^{\ 2})$$

$$T = \frac{2\sigma\gamma}{\tau} + O(\tau^{\ 2}), \qquad U \quad -\sigma\gamma\tau + O(1)$$

$$c(m, n) = -i\tau + O(\tau^{\ 1}), \quad d(m, n) \quad -i\tau + b(m, n) + O(\tau^{\ 1}) \tag{7.24}$$

$$D_{mn} = -i\tau b(m, n) + a^2(m^2 + n^2) + O(\tau^{\ 1})$$

$$iU/d(m, n) = \sigma\gamma + O(1).$$

In the case of perfect conductivity studied in Sections 3 and 4, one source of annoyance was the appearance of $b(m, n)$ as a factor in certain denominators. Here the corresponding term is $-i\tau b(m, n)$ in D_{mn}. Since $b(m, n)$ may become small, or even vanish, we have retained the $a^2(m^2 + n^2)$ term in D_{mn}.

When τ becomes large equations (7.21) become

$$A_{mn}^{(1)} = \frac{2\sigma\gamma a^2 mnP(m - \nu, n)}{a^2(m^2 + n^2) - i\tau b(m, n)} \to 0$$

$$B_{mn}^{(1)} = 2\sigma\gamma P(m - \nu, n)\left[\frac{a^2n^2}{a^2(m^2 + n^2) - i\tau b(m, n)} - 1\right] \tag{7.25}$$

$$\to -2\sigma\gamma P(m - \nu, n)$$

$$C_{mn}^{(1)} = \frac{2\sigma\gamma anP(m - \nu, n)}{b(m, n) + ia^2(m^2 + n^2)/\tau}.$$

When $b(m, n)$ is very small $a^2(m^2 + n^2)$ is nearly equal to $-\sigma^2$ and we may replace the denominator in $C_{mn}^{(1)}$ by

$$b(m, n) + ia^2(m^2 + n^2)/\tau = b(m, n) - i\sigma^2/\tau = b(m, n) + i\beta^2/\tau \tag{7.26}$$

which never vanishes since $b(m, n)$ is either positive real or negative imaginary. Thus the difficulty encountered in Section 3 (and, presumably, also that in Section 4) may be overcome by taking the electrical properties of the reflecting surface into account.

The average value of E_y in the upper medium, from which the average

value of the reflection coefficient may be obtained turns out to be the average value of

$$E^+ + B, \ E \ \nu, 0, z \qquad \exp\{-\sigma\alpha x\}[\exp\{\sigma\gamma z\} + \exp\{-\sigma\gamma z\}(R + B_{\nu 0}^{(2)})]$$

$$d \ \nu, 0 \ B_{\nu 0}^{(2)} \qquad c(\nu, 0 \ h_2 - h \tag{7.27}$$

$$\langle B_{\nu 0}^{(2)} \rangle \quad \frac{2U}{d \ \nu, 0} \sum_l \left\{ \frac{(\sigma \qquad \tau^2}{d \ k, l)} \left[\frac{a^2 l^2}{D_{kl}} - 1\right] \quad i\tau\gamma' \right\} \frac{\pi^2 W \ r - \alpha\beta \ \ s}{L^2}$$

where we have used the relations

$$ic \ \nu, 0 \qquad \tau\gamma', \qquad a\nu \qquad \beta\alpha \qquad \sigma\alpha \ i$$

$$r \qquad ak \qquad 2\pi k/L, \qquad s \qquad al.$$

When we let L approach infinity, the double summation may be replaced by a double integration in the u ual way and we get, after some reduction,

$$\langle B_\nu^{\ 2} \qquad \frac{i2 \quad \sigma^2 \qquad \tau}{\tau\gamma' + \sigma\gamma^{\ 2}} \int_-^\infty dr \int_-^\infty ds \frac{W \ r - \beta\alpha, s}{4} \left[-i\tau\gamma' \right. \tag{7.28}$$

$$\left. + \frac{\sigma \qquad \tau^2)}{c + b} \left(\frac{s^2}{r^2 + s^2 + bc} - 1 \right) \right]$$

where c and b denote functions of r and s defined by

$$ic \qquad (\tau^2 + r^2 + s^2)^{1 \ 2}$$
$$ib \qquad (\sigma^2 + r^2 + s^2)^{1 \ 2} = i(\beta^2 - r^2 - s^2)^{1 \ 2}. \tag{7.29}$$

As g approaches infinity (7.28) should approach the value of its counterpart, given by the double integral in (3.26), which was obtained in Section 3 for reflection from a perfectly conducting but slightly rough surface. That this is the case may be verified with the help of expressions (7.24) which hold for large values of g.

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